Unit. 3
Multiple Integral

Page No.
Date :

Double Integrals:-

1- Find $\int_0^3 \int_0^2 (1+(x-1)^2 y + 4y^2)\,dy\,dx$

$$\int_0^3 \left(y + (x-1)^2 \cdot \frac{y^2}{2} + \frac{4y^3}{3}\right)_0^2 dx$$

$$\int_0^3 \left[\left(2 + (x-1)^2 \cdot \frac{2^2}{2} + \frac{4}{3}\cdot 2^3\right) - (0)\right] dx$$

$$\int_0^3 \left(2 + (x-1)^2 \cdot 2 + \frac{32}{3}\right) dx$$

$$\left(2x + 2\frac{(x-1)^3}{3} + \frac{32}{3}x\right)_0^3$$

$$\left(2\cdot 3 + \frac{2}{3}(3-1)^3 + \frac{32}{3}\cdot 3\right) - \left(\frac{2(0-1)^3}{3}\right)$$

$$\left(6 + \frac{16}{3} + \frac{64}{3}\right) - \left(\frac{-2}{3}\right)$$

$$\Rightarrow \frac{98}{3} + \frac{2}{3} = \frac{100}{3}.$$

2- $\int_0^1 \int_0^{x^2} (x+2y^2)\,dy\,dx$

$$\int_0^1 \left(xy + \frac{2y^3}{3}\right)_0^{x^2} dx$$

$$\int_0^1 \left[\left(x\cdot x^2 + \frac{2(x^2)^3}{3}\right) - 0\right] dx$$

$$\int_0^1 \left(x^3 + \frac{2x^6}{3}\right) dx$$

$$\left(\frac{x^4}{4} + \frac{2}{3}\frac{x^7}{7}\right)_0^1$$

$$\left(\frac{1}{4} + \frac{2}{21}\right) - (0) = \frac{29}{84}.$$

3. Show that

$$\int_1^a \int_1^b \frac{1}{xy}\, dy\, dx = (\log a)\cdot(\log b)$$

LHS:- $\displaystyle\int_1^a \left(\frac{1}{x}\log y\right)_1^b dx$

$$\int_1^a \left[\left(\frac{1}{x}\log b\right) - \left(\frac{1}{x}\log 1\right)\right] dx$$

$\because \log 1 = 0$

$$\int_1^a \left(\frac{1}{x}\log b\right) dx$$

$$\log b \int_1^a \left(\frac{1}{x}\right) dx$$

$$\log b \cdot (\log x)_1^a$$

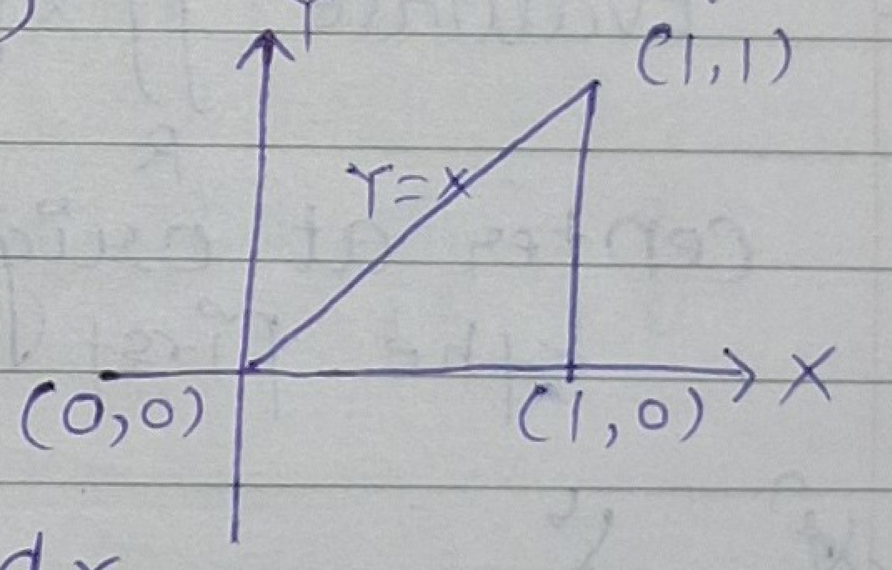
$$\log b\,(\log a - \log 1)$$

$\because \log 1 = 0$

$\log b \cdot \log a \quad = \text{RHS}.$

Hence Proved.

4. Evaluate $\iint_R xy\, dA$ where R is bounded by the triangle whose vertices are $(0,0)$, $(1,0)$ & $(1,1)$



Here $0 \le y \le x$
$0 \le x \le 1$

$$\iint_R xy\, dA = \int_0^1 \int_0^x xy\, dy\, dx$$

$$= \int_0^1 \left(\frac{xy^2}{2}\right)_0^x dx$$

$$\Rightarrow \int_0^1 \frac{x}{2}(x^2 - 0)\, dx = \int_0^1 \frac{x^3}{2} dx$$

$$\Rightarrow \left(\frac{x^4}{8}\right)_0^1 = \frac{1}{8}$$

Double Integral
Multiple Integral

Page No.
Date :

5. $\int_0^{\pi/2}\int_0^1 r\sin x \, dr\, dx = \frac{1}{2}$

LHS:- $\int_0^{\pi/2} \sin x \left(\frac{r^2}{2}\right)_0^1 dx$

$\int_0^{\pi/2} \sin x\left(\frac{1}{2}-0\right) dx$

$\frac{1}{2}\int_0^{\pi/2} \sin x \, dx$

$-\frac{1}{2}\left(\cos x\right)_0^{\pi/2}$

$\frac{-1}{2}\left(\cos \pi/2 - \cos 0\right)$

$\frac{-1}{2}(0-1) = \frac{1}{2}$

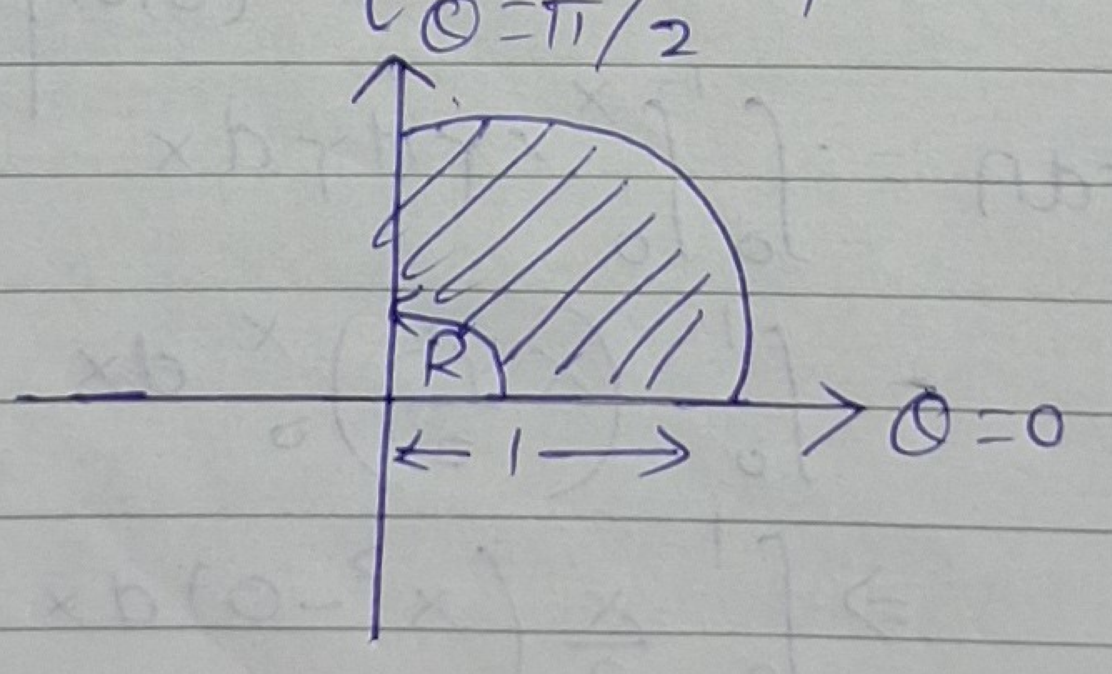
Double Integral in Polar Coordinates

Replace $x = r\cos\theta$

$y = r\sin\theta$

& $dx\,dy = r\,dr\,d\theta$

Q1:- Evaluate $\iint_R xy \, dx\, dy$ where R is the region of the circle with center at origin & radius 1 that lies in the first quadrant.



Substitute

$x = r\cos\theta$

$y = r\sin\theta$

& $dx\,dy = r\,dr\,d\theta$.

$$\iint_R xy\,dx\,dy = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^1 \cos\theta\cdot\sin\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\cos\theta\cdot\sin\theta\,d\theta.$$

$$= \frac{1}{4}\cdot\frac{2}{2}\int_0^{\pi/2}\cos\theta\cdot\sin\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\pi/2}\sin 2\theta\,d\theta$$

$$= \frac{1}{8}\left(\frac{-\cos 2\theta}{2}\right)_0^{\pi/2}$$

$$= \frac{-1}{16}\left(\cos 2(\pi/2) - \cos 0\right)$$

$$= \frac{-1}{16}(-1-1)$$

$$= \frac{(-1)}{16}(-2) = \frac{2}{16} = \frac{1}{8}$$

Page No.
Date :

Q2. Find $\iint_R e^{-(x^2+y^2)}\,dx\,dy$ where R is the region between two circles $x^2+y^2=1$ and $x^2+y^2=4$

$x^2+y^2=1$ $\quad x^2+y^2=2^2$ ($\therefore$ radius $=2$)

Put $x=r\cos\theta$, $y=r\sin\theta$

$0\le\theta\le 2\pi$

$1\le r\le 2$

$$\int_0^{2\pi}\int_1^2 e^{-(r^2\cos^2\theta+r^2\sin^2\theta)}\,r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_1^2 e^{-r^2(\cos^2\theta+\sin^2\theta)}\,r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_1^2 e^{-r^2}\cdot r\,dr\,d\theta \quad (\because \cos^2\theta+\sin^2\theta=1)$$

Take $r^2=t$

$2r\,dr=dt$

$\therefore r\,dr=\frac{dt}{2}$

when $r=1$, $t=1^2=1$

$r=2$, $t=2^2=4$

$$\int_0^{2\pi}\int_1^4 e^{-t}\frac{dt}{2}\,d\theta$$

$$\frac{1}{2}\int_0^{2\pi}\int_1^4 e^{-t}\,dt\,d\theta$$

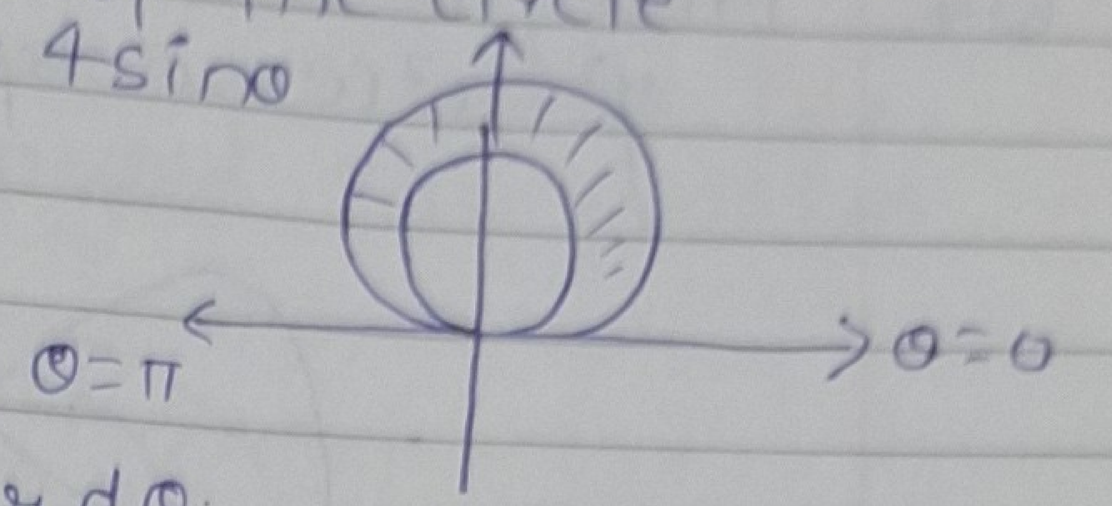
$$\frac{1}{2}\int_0^{2\pi}\left(\frac{e^{-t}}{(-1)}\right)_1^4 d\theta$$

$$\left(\frac{-1}{2}\right)\left(e^{-4}-e^{-1}\right)\int_0^{2\pi}d\theta$$

$$\left(\frac{-1}{2}\right)\left(e^{-4}-e^{-1}\right)(\theta)_0^{2\pi}$$

$$\frac{e^{-1}-e^{-4}}{2}(2\pi)=\pi(e^{-1}-e^{-4})$$

Q3:- Find the area bounded by the circle $r = 2\sin\theta$ & $r = 4\sin\theta$



$$A = \int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi} \left(\frac{r^2}{2}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \left[16\sin^2\theta - 4\sin^2\theta\right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi} (12\sin^2\theta)\, d\theta$$

$$= 6 \int_0^{\pi} \sin^2\theta\, d\theta \qquad \left(\because \cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= 6 \int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

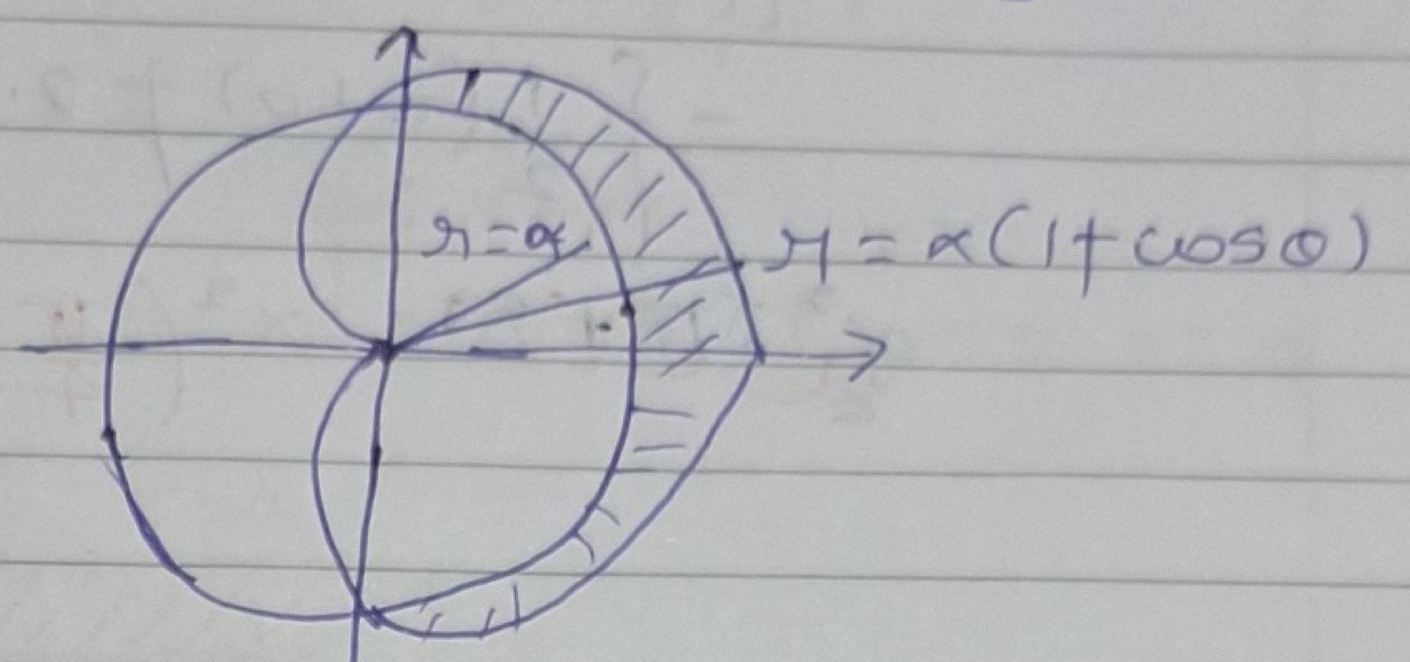
$$= 3 \int_0^{\pi} (1 - \cos 2\theta)\, d\theta$$

$$= 3 \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi}$$

$$= 3\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= 3\pi \qquad (\because \sin 2\pi = 0)$$

Q.5 Find the area that lies inside the cardioids $r = \alpha(1+\cos\theta)$ & outside the circle $r = \alpha$, by double integration



$$\because r = \alpha,\ r = \alpha(1+\cos\theta)$$
$$\alpha = \alpha(1+\cos\theta)$$
$$\alpha = \alpha + \alpha\cos\theta$$
$$\cos\theta = 0$$
$$\theta = \pi/2$$
$$\therefore \theta :- 0 \text{ to } \pi/2$$

$$\text{Area} = 2\int_0^{\pi/2}\int_{\alpha}^{\alpha(1+\cos\theta)} r\,dr\,d\theta$$

$$= \cancel{2\int_0^{\pi/2} (\alpha(1+\cos\theta)}$$

$$= 2\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{\alpha}^{\alpha(1+\cos\theta)} dr\,d\theta$$

$$= \int_0^{\pi/2}\left((\alpha(1+\cos\theta))^2 - \alpha^2\right) d\theta$$

$$= \alpha^2\int_0^{\pi/2}(1+\cos^2\theta + 2\cos\theta - 1)\,d\theta$$

$$= \alpha^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta$$

$$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \alpha^2\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2} + 2\cos\theta\right)d\theta$$

Page No.
Date :

$$\alpha^2 \left[\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + 2\sin\theta\right]_0^{\pi/2}$$

$$\alpha^2 \left[\left\{\frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) + 2\sin(\pi/2)\right\} - \left\{\frac{1}{2}(0+0) + 2\cdot 0\right\}\right]$$

$$\cancel{\frac{\alpha^2}{2}(\pi+8)} \quad \alpha^2\left(\frac{\pi}{4} + 2\right)$$

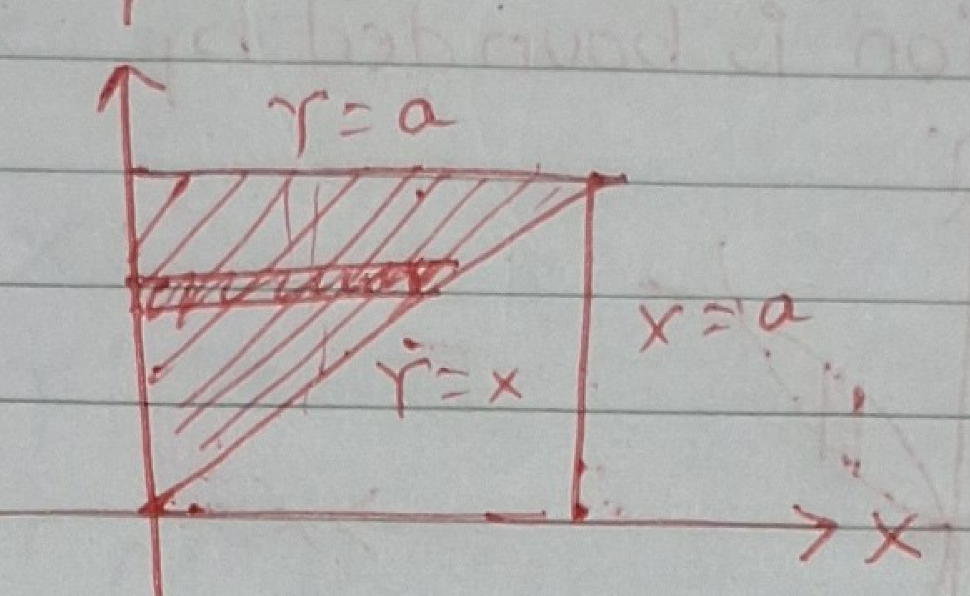
Page No.
Date

Change of Order of Integration:-

→ Change the order & hence evaluate

Q1 $\int_0^a \int_x^a (x^2+y^2)\, dy\, dx$

$y: x \to a$, $y = x,\ y = a$
$x: 0 \to a$, $x = 0,\ x = a$



By changing the order
$x: 0$ to y
$y: 0$ to a

$\int_0^a \int_x^a (x^2+y^2)\, dy\, dx = \int_0^a \int_0^y (x^2+y^2)\, dx\, dy$

$= \int_0^a \left(\frac{x^3}{3} + xy^2\right)_0^y dy$

$= \int_0^a \left(\frac{y^3}{3} + y^3\right) dy$

$= \frac{y^4}{4.3} + \frac{y^4}{4} \Big|_0^a$

$\Rightarrow \frac{a^4}{12} + \frac{a^4}{4} = \frac{a^4}{3}$

Page No.
Date :

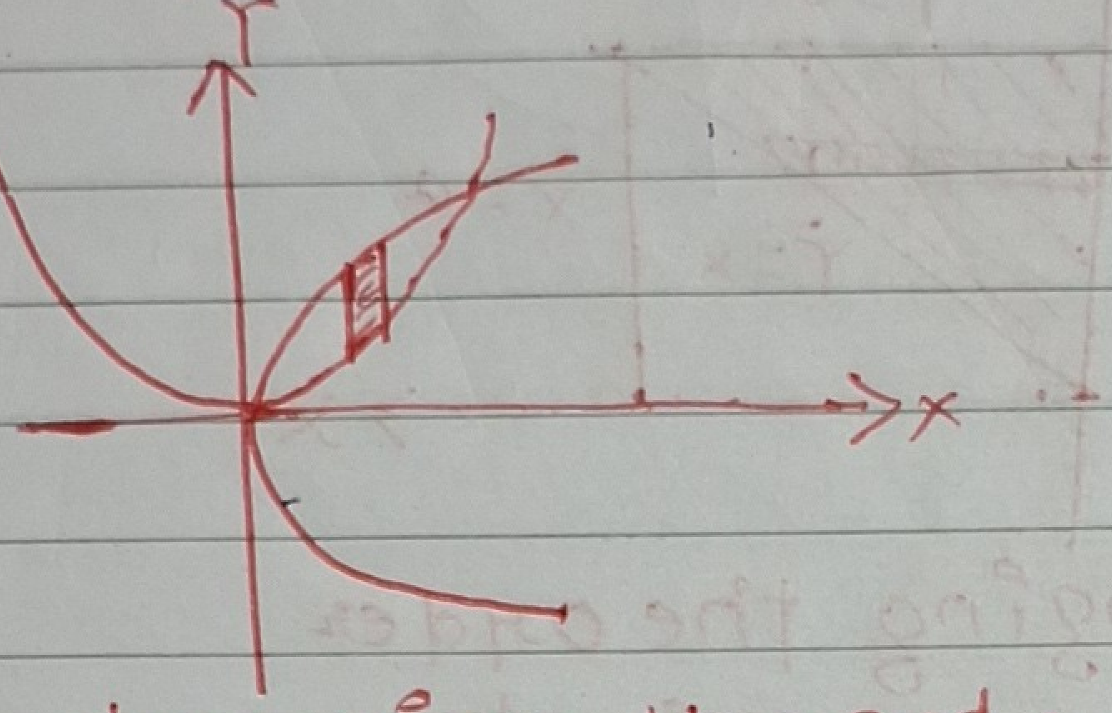
Q2. Change the order of integration

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$$

$$y: \frac{x^2}{4a} \to 2\sqrt{ax}$$

$$x: 0 \to 4a$$

The region is bounded by $x^2 = 4ay$
$y^2 = 4ax$
$x = 0$ & $x = 4a$



By changing the order of Integration we get

$\because x^2 = 4ay \Rightarrow x = 2\sqrt{ay}$

& $y^2 = 4ax \Rightarrow x = y^2/4a$

$$\therefore \left.\begin{aligned} x &: \frac{y^2}{4a} \to 2\sqrt{ay} \\ y &: 0 \to 4a \end{aligned}\right\}$$

$$\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy = \int_0^{4a}\left[\frac{x^2}{2}y\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left\{\frac{(2\sqrt{ay})^2 y}{2} - \frac{y^2}{4a}\cdot\frac{2y}{2}\right\} dy$$

$$= \left[\frac{4ay^3}{6} - \frac{y^6}{192a^2}\right]_0^{4a}$$

$$\Rightarrow \frac{4a(4a)^3}{6} - \frac{(4a)^6}{192a^2} = \frac{128a^4}{3} - \frac{4096a^4}{192}$$

$$= \frac{64a^4}{3}$$

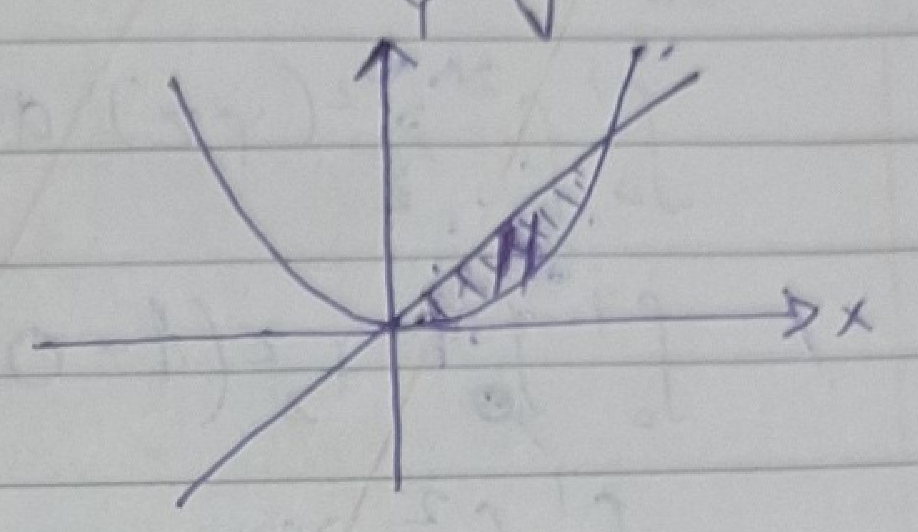
Page No.

Date

Q3 Evaluate $\int_0^1 \int_{x^2}^{x} xy(x+y)\, dy\, dx$ by changing the order of integration.

$y : x^2 \to x$

$x : 0 \to 1$



After changing the order,

$x : y \to \sqrt{y}$

$y : 0 \to 1$

$\int_0^1 \int_y^{\sqrt{y}} xy(x+y)\, dx\, dy$

$\int_0^1 \int_y^{\sqrt{y}} (x^2y + xy^2)\, dx\, dy$

$\int_0^1 \left(\frac{x^3}{3}y + \frac{x^2}{2}y^2 \right)_y^{\sqrt{y}} dy$

$\int_0^1 \left\{ \left(\frac{y^2\sqrt{y}}{3} + \frac{y^3}{2} \right) - \left(\frac{y^4}{3} + \frac{y^4}{2} \right) \right\} dy$

$\int_0^1 \left(\frac{y^{5/2}}{3} + \frac{y^3}{2} - \frac{5y^4}{6} \right) dy$

$\left[\frac{y^{7/2}}{3(7/2)} + \frac{1}{2}\frac{y^4}{4} - \frac{5}{6}\frac{y^5}{5} \right]_0^1$

$\left(\frac{2}{21} + \frac{1}{8} - \frac{1}{6} \right) = \frac{3}{56}$

Triple Integral

Evaluate:-

Q1:- $\int_0^1\int_0^2\int_0^1 xyz\,dx\,dy\,dz$

$\int_0^1\int_0^2 yz\left(\frac{x^2}{2}\right)_0^1 dy\,dz$

$\int_0^1\int_0^2 \frac{yz}{2}(1-0)\,dy\,dz$

$\int_0^1 \left(\frac{z}{2}\right)\left(\frac{y^2}{2}\right)_0^2 dz$

$\int_0^1 \frac{z}{4}(4-0)\,dz$

$\int_0^1 z\,dz$

$\left(\frac{z^2}{2}\right)_0^1 = \left(\frac{1}{2}-0\right) = \frac{1}{2}$

Q2:- $\int_0^1\int_0^x\int_0^{x+y}(2x+y-1)\,dz\,dy\,dx$

$\int_0^1\int_0^x (2x+y-1)(z)_0^{x+y}\,dy\,dx$

$\int_0^1\int_0^x (2x+y-1)(x+y-0)\,dy\,dx$

$\int_0^1\int_0^x (2x^2+2xy+xy+y^2-x-y)\,dy\,dx$

$\int_0^1\int_0^x (2x^2+3xy+y^2-x-y)\,dy\,dx$

$\int_0^1 \left(2x^2y+\frac{3x}{2}y^2+\frac{y^3}{3}-xy-\frac{y^2}{2}\right)_0^x dx$

$\int_0^1 \left(2x^3+\frac{3}{2}x^3+\frac{x^3}{3}-x^2-\frac{x^2}{2}\right)dx$

$$\int_0^1 \left(2x^3 - x^2 + \frac{5x^3}{6} - \frac{x^2}{2}\right) dx$$

$$= \frac{5}{24}$$

Q3:- $\displaystyle\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^6 \frac{dz\,dy\,dx}{\sqrt{x^2+y^2}}$

Put $x = r\cos\theta$ & $y = r\sin\theta$

$dy\,dx = r\,dr\,d\theta$

since $0 \le y \le \sqrt{25-x^2}$

$y = \sqrt{25-x^2}$

$y^2 = 25 - x^2$

$x^2 + y^2 = 25 = 5^2$

$\Rightarrow r = 5$

$\therefore 0 \le r \le 5$

$0 \le \theta \le \pi/2$

$$\int_0^{\pi/2}\int_0^5\int_0^6 \frac{\cancel{r}\,dz\,dr\,d\theta}{\cancel{r}}$$

$$\left\{\begin{aligned} &\sqrt{x^2+y^2} \\ &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{r^2} \\ &= r \end{aligned}\right.$$

$$\int_0^{\pi/2}\int_0^5\int_0^6 dz\,dr\,d\theta$$

$$\int_0^{\pi/2}\int_0^5 (z)_0^6\,dr\,d\theta$$

$$\int_0^{\pi/2}\int_0^5 (6-0)\,dr\,d\theta$$

$$\int_0^{\pi/2} 6(r)_0^5\,d\theta$$

$$\int_0^{\pi/2} 30\,d\theta = 30(\theta)_0^{\pi/2} = 30\cdot\pi/2$$

$$= 15\pi$$

Q4 - $\int_0^2 \int_1^2 \int_1^4 e^{x+z}\, dz\, dy\, dx$

$\int_0^2 \int_1^2 \int_1^4 e^x . e^z\, dz\, dy\, dx$

$\int_0^2 \int_1^2 e^x . (e^z)_1^4\, dy\, dx$

$\int_0^2 \int_1^2 e^x (e^4 - e^1)\, dy\, dx$

$(e^4 - e) \int_0^2 e^x (y)_1^2\, dx$

$(e^4 - e) \int_0^2 e^x (2-1)\, dx$

$(e^4 - e) \int_0^2 e^x\, dx$

$(e^4 - e)(e^2 - 1)$

Q5

Q.5 Evaluate $\int_2^3\int_{-1}^4\int_0^1 (4x^2y - z^3)\,dz\,dy\,dx$

$$= \int_2^3\int_{-1}^4 \left(4x^2yz - \frac{z^4}{4}\right)_0^1 dy\,dx$$

$$= \int_2^3\int_{-1}^4 \left(4x^2y - \frac{1}{4}\right) dy\,dx$$

$$= \int_2^3 \left(4x^2\frac{y^2}{2} - \frac{1}{4}y\right)_{-1}^4 dx$$

$$= \int_2^3 \left\{\left(\frac{4x^2}{2}(16) - \frac{1}{4}(4)\right) - \left(\frac{4x^2}{2} + \frac{1}{4}\right)\right\} dx$$

$$= \int_2^3 \left\{32x^2 - 1 - 2x^2 - \frac{1}{4}\right\} dx$$

$$= \int_2^3 \left\{30x^2 - \frac{5}{4}\right\} dx$$

$$= \cancel{\left(30x^2 z - \frac{15}{4}z\right)_2^3}$$

$$= \left(\frac{30x^3}{3} - \frac{5}{4}x\right)_2^3$$

$$= \left(10x^3 - \frac{5x}{4}\right)_0^3$$

$$= \left(270 - \frac{15}{4}\right) - (0)$$

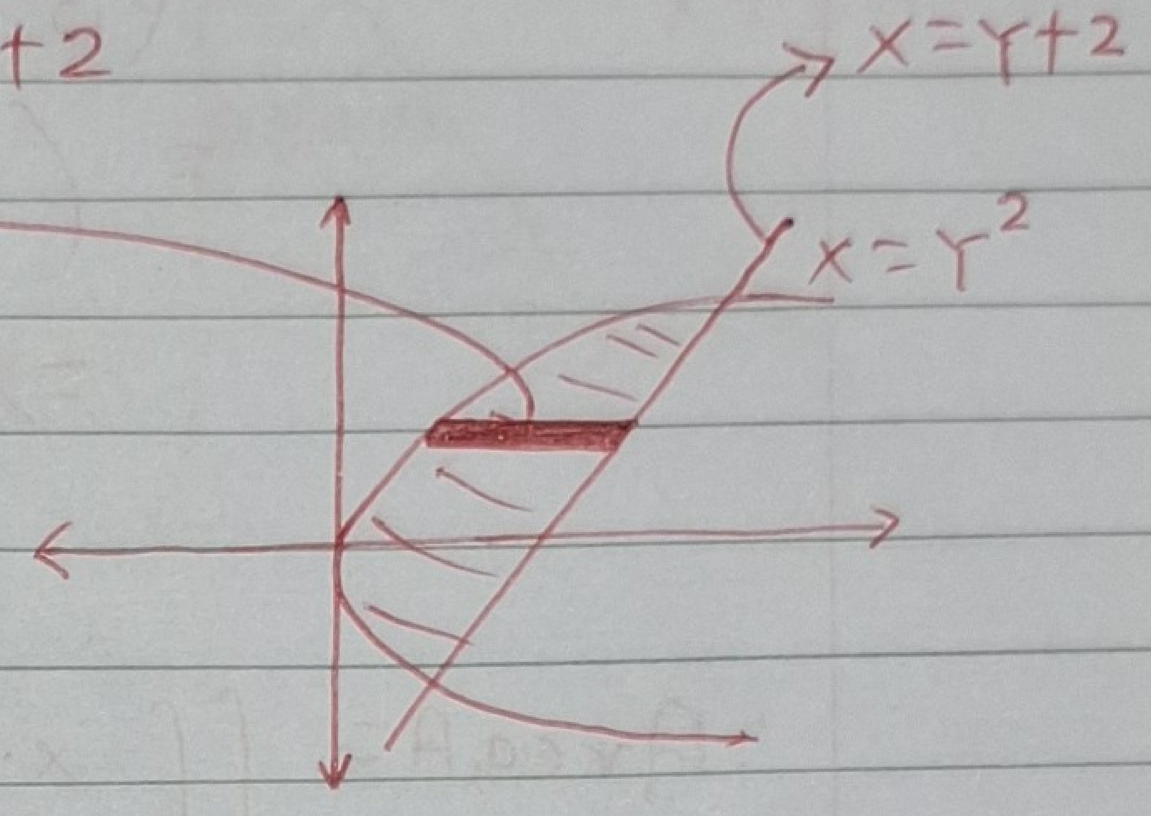
$$= \frac{1080-15}{4}$$

$$= \frac{1065}{4}.$$

Application of Multiple Integrals

To find Area:-

1- Evaluate $\iint_D x \cdot y \, dA$, D is the region bounded by the curves $x = y^2$ & $x = y+2$



x :- y^2 to $y+2$

To find limit of y :-

Solve) -

$y^2 = y+2$

$y^2 - y - 2 = 0$

$y^2 - 2y + y - 2 = 0$

$y(y-2) + 1(y-2) = 0$

$(y+1)(y-2) = 0$

$y = -1, \; y = 2$

$\therefore$ y : -1 to 2

$$\therefore \text{Area} = \iint_D xy \, dx \, dy$$

$$= \int_{-1}^{2} \int_{y^2}^{y+2} x \cdot y \, dx \, dy$$

$$= \int_{-1}^{2} \left(y \frac{x^2}{2} \right)_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2} \frac{y}{2} \left[(y+2)^2 - y^2 \right] dy$$

$$= \int_{-1}^{2} \frac{y}{2} \left[\cancel{y^2} + 4 + 4y - \cancel{y^2} \right] dy$$

$$= \int_{-1}^{2} \frac{y}{2} \left[4y + 4 \right] dy$$

$$= \int_{-1}^{2} \frac{1}{2} \left(4y^2 + 4y \right) dy$$

$$= \int_{-1}^{2} (2y^2 + 2y)\,dy$$

$$= \left(\frac{2y^3}{3} + \frac{2y^2}{2}\right)_{-1}^{2}$$

$$= \left(\frac{2}{3}(2)^3 + (2)^2\right) - \left(\frac{2}{3}(-1)^3 + (-1)^2\right)$$

$$= \left(\frac{16}{3} + 4\right) - \left(\frac{-2}{3} + 1\right)$$

$$\Rightarrow \frac{28}{3} - \frac{1}{3} = \frac{27}{3}$$

$\therefore$ Area, $A = \iint_D x \cdot y \, dx\,dy$

$$= \int_{-1}^{2} \int_{y^2}^{y+2} x \cdot y \, dx\,dy$$

$$= \int_{-1}^{2} \left(\frac{x^2}{2} y\right)_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2} \frac{1}{2} \left(y(y+2)^2 - y(y^2)^2\right) dy$$

$$= \int_{-1}^{2} \frac{1}{2} \left(y(y^2 + 4y + 4) - y^5\right) dy$$

$$= \int_{-1}^{2} \frac{1}{2} \left(y^3 + 4y^2 + 4y - y^5\right) dy$$

$$= \frac{1}{2} \left[\frac{y^4}{4} + \frac{4y^3}{3} + \frac{4y^2}{2} - \frac{y^6}{6}\right]_{-1}^{2}$$

$$= \frac{1}{2} \left[\left\{\frac{2^4}{4} + \frac{4}{3}(2)^3 + \frac{4}{2}(2)^2 - \frac{2^6}{6}\right\} - \left\{\frac{(-1)^4}{4} + \frac{4}{3}(-1)^3 + \frac{4}{2}(-1)^2 - \frac{(-1)^6}{6}\right\}\right]$$

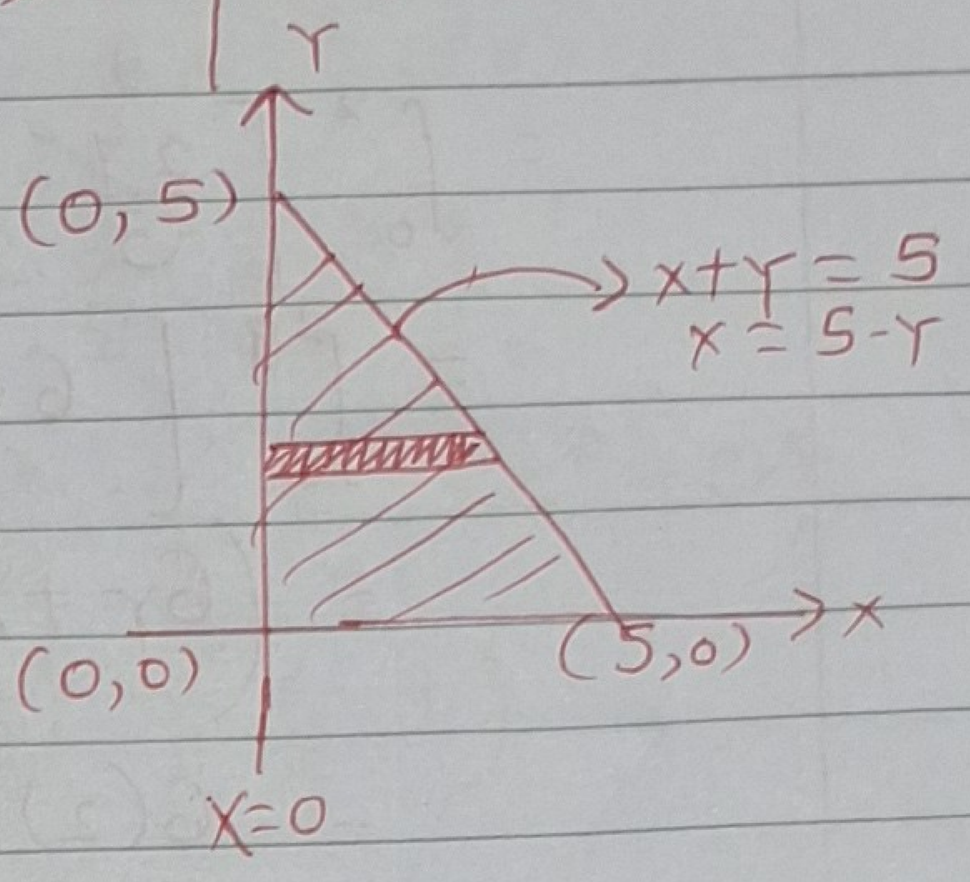
$$= \frac{1}{2} \left[\left\{4 + \frac{32}{3} + 8 - \frac{64}{6}\right\} - \left\{\frac{1}{4} - \frac{4}{3} + 2 - \frac{1}{6}\right\}\right]$$

$$A = \frac{135}{24}$$

Page No.
Date:

2- Find the area of triangle whose vertices are (0,0), (5,0) + (0,5)

x: 0 to $5-y$
y: 0 to 5



$$\therefore \text{Area} = \int_0^5 \int_0^{5-y} dx\,dy$$

$$= \int_0^5 (5-y-0)\,dy$$

$$= \int_0^5 (5-y)\,dy$$

$$= \left(5y - \frac{y^2}{2}\right)_0^5$$

$$= 5(5) - \frac{5^2}{2}$$

$$\Rightarrow 25 - \frac{25}{2} = \frac{25}{2}$$

3- Evaluate $\iint_R (1+(x-1)^2+4y^2)\,dA$, where R be the region given as $R=[0,3]\times[0,2]$.

$0 \le x \le 3$ $\quad$ $0 \le y \le 2$

$$\int_0^2\int_0^3 (1+(x-1)^2+4y^2)\,dx\,dy$$

$$\int_0^2\int_0^3 (1+x^2+1-2x+4y^2)\,dx\,dy$$

$$\int_0^2\int_0^3 (x^2-2x+2+4y^2)\,dx\,dy$$

Page No.
Date :

$$= \int_0^2 \left(\frac{x^3}{3} - \frac{2x^2}{2} + 2x + 4xy^2\right)_0^3 dy$$

$$= \int_0^2 \left[\cancel{\frac{27}{3}}^{9} - \cancel{9} + 6 + 12y^2\right] dy$$

$$= \int_0^2 \left[6 + 12y^2\right] dy$$

$$= \left(6y + \frac{12y^3}{3}\right)_0^2$$

$$= 6(2) + 4(2)^3$$

$$= 12 + 32$$

Area $= 44$.

To Find Volume :-

1. Find the volume of the solid bounded by $0 \leq x \leq 1,\ 0 \leq y \leq x,\ x+y \leq z \leq e^{x+y}$.

Volume of solid $= \int_0^1 \int_0^x \int_{x+y}^{e^{x+y}} dz\,dy\,dx$

$$= \int_0^1 \int_0^x (z)_{x+y}^{e^{x+y}}\, dy\,dx$$

$$= \int_0^1 \int_0^x \left(e^{x+y} - x - y\right) dy\,dx$$

$$= \int_0^1 \left(e^{x+y} - xy - \frac{y^2}{2}\right)_0^x dx$$

$$= \int_0^1 \left\{\left(e^{x+x} - x \cdot x - \frac{x^2}{2}\right) - \left(e^{x+0} - 0 \cdot x - \frac{0^2}{2}\right)\right\} dx$$

Page No.
Date :

$$= \int_0^1 \left\{ \left(e^{2x} - x^2 - \frac{x^2}{2} \right) - \left(e^x \right) \right\} dx$$

$$= \int_0^1 \left(e^{2x} - e^x - \frac{3}{2}x^2 \right) dx$$

$$= \left(\frac{e^{2x}}{2} - e^x - \frac{3}{2}\left(\frac{x^3}{3}\right) \right)_0^1$$

$$= \left(\frac{e^2}{2} - e^1 - \frac{1}{2} \right) - \left(\frac{e^0}{2} - e^0 - 0 \right)$$

$$= \left(\frac{e^2}{2} - e - \frac{1}{2} \right) - \left(\frac{1}{2} - 1 \right)$$

$$= \frac{e^2}{2} - e - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}$$

Volume $= \frac{e^2}{2} - e$

2. Find the volume of the solid bounded by the four planes:

$x = 0,\ y = 0,\ z = x + y,\ z = 1 - x - y$

$$x + y = 1 - x - y$$

$$2x + 2y = 1$$

$$y = \frac{1}{2} - x$$

$$\left.\begin{aligned} 0 &\le y \le \frac{1}{2} - x \\ 0 &\le x \le \frac{1}{2} \\ x + y &\le z \le 1 - x - y \end{aligned}\right\}$$

$$\therefore \text{volume}, V = \int_0^{1/2}\int_0^{1/2-x}\int_{x+y}^{1-x-y} 1 \cdot dz\,dy\,dx$$

$$V = \int_0^{1/2}\int_0^{1/2-x} [z]_{x+y}^{1-x-y}\, dy\,dx$$

$$V = \int_0^{1/2}\int_0^{1/2-x} (1-x-y-x-y)\, dy\,dx$$

$$V = \int_0^{1/2}\int_0^{1/2-x} (1-2x-2y)\, dy\,dx$$

$$V = \int_0^{1/2} \left(y - 2xy - 2\frac{y^2}{2} \right)_0^{1/2-x} dx$$

$$V = \int_0^{1/2} \left\{ \left(\frac{1}{2}-x\right) - 2x\left(\frac{1}{2}-x\right) - \left(\frac{1}{2}-x\right)^2 \right\}_0^{\frac{1}{2}-x} dx$$

$$V = \int_0^{1/2} \left\{ \frac{1}{2} - x - \cancel{x} + 2x^2 - \frac{1}{4} - x^2 + \cancel{x} \right\} dx$$

$$V = \int_0^{1/2} \left\{ \frac{1}{4} - x + x^2 \right\} dx$$

$$V = \left(\frac{1}{4}x - \frac{x^2}{2} + \frac{x^3}{3} \right)_0^{\frac{1}{2}}$$

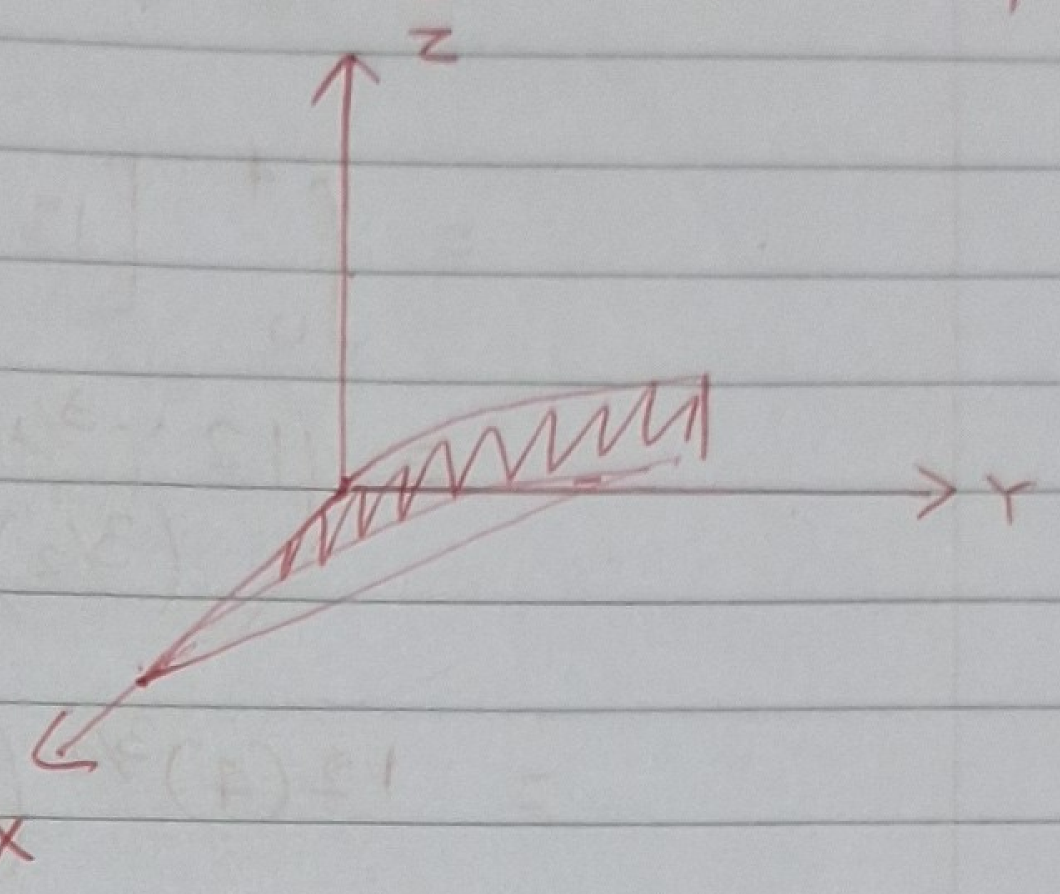
$$V = \frac{1}{4}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3$$

$$= \cancel{\frac{1}{8}} - \cancel{\frac{1}{8}} + \frac{1}{24}$$

$$V = \frac{1}{24}$$

3. Determine the volume of the region that lies behind the plane $x+y+z=8$ & in front of the region in the yz-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ & $z=\frac{3}{4}y$

equate:- $\frac{3}{2}\sqrt{y}=\frac{3}{4}y$



$\frac{9}{4}y=\frac{9}{16}y^2$

$y^2=4y$

$y^2-4y=0$

$y(y-4)=0$

$y=0,\ y=4$

$\therefore y: 0$ to 4

$z: \frac{3}{4}y$ to $\frac{3}{2}\sqrt{y}$ $\quad \because x+y+z=8$

$x=8-y-z$

$\therefore x: 0$ to $8-y-z$

$\therefore$ Volume $=\iiint_R dv$

$$=\int_0^4\int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}}\int_0^{8-y-z} dx\,dz\,dy$$

$$=\int_0^4\int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}}(8-y-z-0)\,dz\,dy$$

$$=\int_0^4\left(8z-yz-\frac{z^2}{2}\right)_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}}$$

$$=\int_0^4\left\{\left(8\cdot\frac{3}{2}\sqrt{y}-y\cdot\frac{3}{2}\sqrt{y}-\frac{\left(\frac{3}{2}\sqrt{y}\right)^2}{2}\right)-\left(8\cdot\frac{3}{4}y-y\cdot\frac{3}{4}y-\frac{1}{2}\left(\frac{3}{4}y\right)^2\right)\right\}$$

Page No.
Date :

$$= \int_0^4 \left\{ \left(\frac{24}{2}\sqrt{r} - \frac{3}{2} r^{3/2} - \frac{9}{4} r \right) - \left(\frac{24}{4} r - \frac{3}{4} r^2 - \frac{9}{16} r^2 \right) \right\} dr$$

$$= \int_0^4 \left[12\sqrt{r} - \frac{3}{2} r^{3/2} - \frac{9}{4} r - 6r - \frac{3}{4} r^2 - \frac{9}{16} r^2 \right] dr$$

$$= \int_0^4 \left[12\sqrt{r} - \frac{3}{2} r^{3/2} - \frac{33}{4} r + \frac{21}{16} r^2 \right] dr$$

$$= \left(12 \cdot \frac{r^{3/2}}{(3/2)} - \frac{3}{2} \frac{r^{5/2}}{(5/2)} - \frac{33}{4} \frac{r^2}{2} + \frac{21}{6} \frac{r^3}{3} \right)_0^4$$

$$= 12(4)^{3/2}\left(\frac{2}{3}\right) - \frac{3}{5}(4)^{5/2} - \frac{33}{8}(4)^2 + \frac{7}{16}(4)^3$$

$\therefore$ Volume $= \frac{49}{5}$.