Assignment 2
Unit- 1.

Laplace Transform

Let f be a real (complex) valued function of variable t such that $t \geq 0$, then the Laplace transform of f. is denoted as $L[f(t)]$ & it is given as

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = F(s)$$

Whenever the above Integral converges then only the laplace transform of 'f' exists else not.

Ex:- If $f(t) = 3$ for $t \geq 0$
then $L[f(t)] = ?$

$$\because L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[3] = \int_0^\infty e^{-st} (3)\,dt$$

$$= 3\left(\frac{e^{-st}}{-s}\right)_0^\infty$$

$$= 3\left(\frac{e^{-\infty}}{-s} - \frac{e^0}{-s}\right)$$

$$= 3\left(\frac{1}{s}\right)$$

$$\boxed{L[(3)] = \frac{3}{s}}$$

Ex:- Let f be a function defined as
f(t) = t for t ≥ 0, then find L[f(t)]

$$L[f(t)] = \int_0^\infty \underset{'v'}{e^{-st}} \cdot \underset{'u'}{f(t)}\,dt$$

$$= \int_0^\infty e^{-st} \cdot t\,dt$$

Apply By parts

$$= \left[t\frac{e^{-st}}{(-s)} - (1)\frac{e^{-st}}{s^2} \right]_0^\infty$$

$$\therefore L[f(t)] = \left\{ e^{-\infty} - e^{-\infty} \right\} - \left\{ 0 - \frac{e^0}{s^2} \right\}$$

$$L[f(t)] = \frac{1}{s^2} \quad \left(\because e^{-\infty} = 0,\ e^0 = 1 \right)$$

Properties of Laplace Transform:-

1- Linearity Property:-

Let $L[f(t)] = F(s)$ & $L[g(t)] = G(s)$
then $L[\alpha f(t) + \beta g(t)]$
$= \alpha L[f(t)] + \beta L[g(t)]$
Where α, β are constants.

Page No.
Date :

2- First shifting theorem

If $L[f(t)] = F(s)$

then $L[e^{at} \cdot f(t)] = F(s-a)$
where a is any constant

3- Second shifting theorem

If f & g be functions such that
$L[f(t)] = F(s)$
& $g(t) = f(t-a)$, $t > a$
0 , $t < a$
then $L[g(t)] = e^{-as} \cdot F(s)$

4- Change of scale property

If $L[f(t)] = F(s)$
then $L[f(at)] = \frac{1}{a} \cdot F\left(\frac{s}{a}\right)$, $a \neq 0$

5- Laplace transform of Derivatives

If $L[f(t)] = F(s)$
then $L[f'(t)] = \cancel{s \cdot F(s) - f(0)}$
$= s \cdot F(s) - f(0)$

& $L[f''(t)] = s^2 \cdot F(s) - sf(0) - f'(0)$

6- Laplace transform of Integrals

If $L[f(t)] = F(s)$

then $L\left[\int_0^t f(\alpha)\,d\alpha\right] = \frac{F(s)}{s}$

7- Multiplication by t^n

If $L[f(t)] = F(s)$

then

$$L[t^n \cdot f(t)] = (-1)^n \cdot \frac{d^n}{ds^n}(F(s))$$

Q- Find the Laplace transform of the following functions

1- $L[5t^2 + 4e^{-t}]$

⇒ By Linearity

$L[5t^2] + L[4e^{-t}]$

$5 \cdot L[t^2] + 4\,L[e^{-t}]$

$5 \cdot \frac{(2!)}{s^{2+1}} + 4\,\frac{1}{s-(-1)}$

$\Rightarrow \frac{10}{s^3} + \frac{4}{s+1}$

2- $L[t^3 + 5t - 2]$

$L[t^3] + 5L[t] - 2L[1]$

$\frac{3!}{s^{3+1}} + 5 \cdot \frac{1}{s^2} - 2 \cdot \frac{1}{s}$

$\Rightarrow \frac{6}{s^4} + \frac{5}{s^2} - \frac{2}{s}$

Formulae

$L[1] = \frac{1}{s}$ $\quad s>0$

$L[t] = \frac{1}{s^2}$ $\quad s>0$

$L[t^n] = \frac{n!}{s^{n+1}}$ $\quad s>0$

$L[e^{at}] = \frac{1}{s-a}$ $\quad s>0$

$L[\sin at] = \frac{a}{s^2+a^2}$ $\quad s>0$

$L[\cos at] = \frac{s}{s^2+a^2}$ $\quad s>0$

$L[\sinh at] = \frac{a}{s^2-a^2}$ $\quad , s>|a|$

$L[\cosh at] = \frac{s}{s^2-a^2}$ $\quad , s>|a|$

Q:- Find the Laplace transform of the following.

(a) $f(t) = 6t^3 - 3\cos(2t) + 7e^{-2t}$

By linearity

$6L[t^3] - 3L[\cos 2t] + 7\,L[e^{-2t}]$

$6 \cdot \frac{(3!)}{s^{3+1}} - 3 \cdot \frac{s}{s^2-4} + 7 \cdot \frac{1}{s-(-2)}$

$\Rightarrow \frac{36}{s^4} - \frac{3s}{s^2-4} + \frac{7}{s+2}$

(b) $f(t) = e^{-2t}\cos(2t)$

By First shifting property,

$$F(s) = L[\cos 2t] = \frac{s}{s^2+4}$$

Replace $s \to s+2$ ($\because$ we have $e^{-2t} \Rightarrow a=-2$)

$$\therefore L[f(t)] = \frac{(s+2)}{(s+2)^2+4}$$

(c) $L\left[\int_0^t \sin 3x\, dx\right] = ?$

$$\therefore f(t) = \sin 3t$$

$$L[f(t)] = \frac{3}{s^2+9} = F(s)$$

$$\therefore L\left[\int_0^t \sin 3x\, dx\right] = \frac{F(s)}{s} = \frac{3}{s(s^2+9)}$$

Q Find laplace transform of $f(t) = [\cos(t)]^2$

$$\because \cos 2t = 2\cos^2 t - 1$$

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$\therefore L[(\cos t)^2] = L\left(\frac{1+\cos 2t}{2}\right)$$

$$= L\left(\frac{1}{2}\right) + L\left(\frac{\cos 2t}{2}\right)$$

$$= \frac{1}{2}L(1) + \frac{1}{2}L(\cos 2t)$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{2}\cdot\frac{s}{s^2+4}$$

Page No.
Date :

Q: $L[t\sin t] = ?$

$\because L(\sin t) = \frac{1}{s^2+1} = F(s)$

$n=1$

$\therefore L[t^1 \sin t] = (-1)^n \frac{d^n}{ds^n} F(s)$

Here $n=1$

$= (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$= (-1)(-1)(s^2+1)^{-2}(2s)$

$= 2s(s^2+1)^{-2}$

$= \frac{2s}{(s^2+1)^2}$

Q. $L[t^2 \sin t] = ?$

$F(s) = L(\sin t) = \frac{1}{s^2+1}$

$\therefore L[t^2 \sin t] = (-1)^2 \frac{d^2}{ds^2} F(s)$

Here $n=2$.

$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)$

$= \frac{d}{ds}\left((-1)(s^2+1)^{-2}(2s)\right)$

$= (-1)\left[(-2)(s^2+1)^{-3}(4s^2) + 2(s^2+1)^{-2}\right]$

$= 8s^2(s^2+1)^{-3} - 2(s^2+1)^{-2}$.

$\cos 2t = 2\cos^2 t - 1$
$\cos 2t = 1 - 2\sin^2 t$

Convolution Integral:-

Let f & g are piecewise continous functions defined for $t \geq 0$, then their convolution integral is denoted by $(f * g)(t)$.

where
$$(f * g) = \int_0^t f(x) \cdot g(t-x)\,dx$$

Convolution Theorem:-

Let $f(t)$ & $g(t)$ are piecewise continous functions for $t \geq 0$ & of exponential order at infinity then

$$L[f * g] = F(s) \cdot G(s)$$

where

$$L[f(t)] = F(s)$$
$$L[g(t)] = G(s)$$

Q:- $f(t) = 0 \qquad 0 \leq t \leq 2$
$\quad = e^{-2t} \qquad t \geq 2$

Find $L[f(t)] = ?$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st}(0)\,dt + \int_2^\infty e^{-st} \cdot e^{-2t}\,dt$$

$$= 0 + \int_2^\infty e^{-t(2+s)}\,dt$$

$$= \left[\frac{e^{-t(2+s)}}{2-s}\right]_2^\infty$$

$$= \left(0 - \frac{e^{-2(2+s)}}{2-s}\right)$$

Inverse Laplace Transform

$L^{-1}(1/s) = 1$

$L^{-1}(1/s^2) = t$

$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$

$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh(at)$

$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh(at)$

1- Linearity property :-

Let $L^{-1}[F_1(s)] = F_1(t)$

$L^{-1}[F_2(s)] = F_2(t)$

C_1, C_2 are constants

then

$L^{-1}[C_1 F_1(s) + C_2 F_2(s)]$.

$= C_1 L^{-1}[F_1(s)] + C_2 L^{-1}[F_2(s)]$

2. First Shifting Property

$$L^{-1}[F(s)] = f(t)$$
then
$$L^{-1}[F(s-a)] = e^{at} f(t)$$
or $L^{-1}[F(s-a)] = e^{at} \cdot L^{-1}[F(s)]$

3. Second shifting Property

$L^{-1}[F(s)] = f(t)$ then
$$L^{-1}[e^{-as} F(s)] = f(t-a) \quad \text{if } t>a$$
$$= 0 \quad t<a$$

4. Change of scale Property

$L^{-1}[F(s)] = f(t)$
then $L^{-1}[F(ks)] = \frac{1}{k} \cdot f(t/k)$

5. $L^{-1}[F(s)] = f(t)$
$$L^{-1}[F^{(n)}(s)] = (-1)^n \cdot t^n \cdot f(t)$$

6. $$L^{-1}\left[\int_s^{\infty} F(u)\,du\right] = f(t)/t$$

7. $$L^{-1}[s^n \cdot F(s)] = f^{(n)}(t)$$

8. $L^{-1}[F(s)] = f(t)$
$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\,du.$$

9. Convolution Theorem
$$L^{-1}[F(s) * G(s)] = \int_0^t f(u) \cdot g(t-u)\,du$$
$$= f * g$$

Examples

Q. Find the Inverse Laplace Transform of following using partial fraction method

1- $F(s) = \frac{s+1}{s^2-4} = \frac{s+1}{(s+2)(s-2)} = \frac{A}{s+2} + \frac{B}{s-2}$

$s+1 = A(s-2) + B(s+2)$

$A = 1/4, \; B = 3/4$

$\therefore F(s) = \frac{1}{4(s+2)} + \frac{3}{4(s-2)}$

$L^{-1}[F(s)] = \frac{1}{4} L^{-1}\left(\frac{1}{s+2}\right) + \frac{3}{4} L^{-1}\left(\frac{1}{s-2}\right)$

$L^{-1}[F(s)] = \frac{1}{4} e^{-2t} + \frac{3}{4} e^{2t}$

2- $F(s) = \frac{1}{s^2-s-2} = \frac{1}{s^2-2s+s-2}$

$= \frac{1}{s(s-2)+1(s-2)}$

$F(s) = \frac{1}{(s+1)(s-2)}$

$F(s) = \frac{A}{s+1} + \frac{B}{s-2}$

$\therefore 1 = A(s-2) + B(s+1)$

$1 = As - 2A + Bs + B$

$B - 2A = 1, \quad A + B = 0$

$B + 2B = 1 \quad A = -B$

$B = 1/3 \quad A = -1/3$

$F(s) = \frac{-1}{3(s+1)} + \frac{1}{3(s-2)}$

Page No.
Date

$\therefore L^{-1}[F(s)] = \frac{-1}{3} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3} L^{-1}\left[\frac{1}{s-2}\right]$

$\therefore L^{-1}[F(s)] = \frac{-1}{3} e^{-t} + \frac{1}{3} e^{2t}$

3. $L^{-1}\left[\frac{1}{(s-6)^3}\right] = ?$

$\because L\left[\frac{n!}{s^{n+1}}\right] = t^n$

$\cancel{L^{-1}\left[\frac{1}{2}\frac{2!}{(s-6)^3}\right]}$

$\cancel{= \frac{1}{2} L^{-1}\left[\frac{2!}{(s-6)^{2+1}}\right]}$

$= \cancel{\frac{1}{2}}$

$L^{-1}\left[\frac{1}{(s-6)^3}\right] = e^{6t} L^{-1}[F(s)]$

$= e^{6t} L^{-1}\left[\frac{1}{s^3}\right]$

$= e^{6t} L^{-1}\left[\frac{1}{2!}\frac{2!}{s^{2+1}}\right]$

$= \frac{e^{6t}}{2} L^{-1}\left[\frac{2!}{s^{2+1}}\right]$

$= \left(\frac{e^{6t}}{2}\right) t^2 = \frac{t^2 e^{6t}}{2}$

4. $L^{-1}\left[\frac{1}{s^2-8s+25}\right] = L^{-1}\left[\frac{1}{s^2-8s+16+9}\right]$

$= L^{-1}\left[\frac{1}{(s-4)^2+3^2}\right]$

$= e^{4t} L^{-1}\left[\frac{1}{s^2+3^2}\right]$

$= e^{4t}\cdot\left(\frac{\sin 3t}{3}\right)$

(5) $L^{-1}\left[\frac{2\cdot e^{-2s}}{s^2+4}\right] = 2L^{-1}\left[\frac{e^{-2s}}{s^2+4}\right]$

$= 2L^{-1}\left[e^{-2s}\cdot\frac{1}{s^2+4}\right]$

$= \cancel{2f(t-2) \text{ if } t>2,\ 0 \text{ if } t<2}$

Here $a=2$, $F(s) = \frac{1}{s^2+4}$

$f(t) = L^{-1}[F(s)]$

$= L^{-1}\left[\frac{1}{s^2+4}\right]$

$= \frac{\sin 2t}{2}$

$\therefore L^{-1}\left[\frac{2e^{-2s}}{s^2+4}\right] = \begin{cases} 2f(t-2) & \text{if } t>2 \\ 0 & t<2 \end{cases}$

$= \frac{2\sin(2(t-2))}{2} = \sin(t-2) \quad t>2$

$= 0 \quad t<2$

(6) $L^{-1}\left[\frac{1}{(s-1)(s+2)^2}\right]$

Take $F(s) = \frac{1}{s-1}$, $G(s) = \frac{1}{(s+2)^2}$

$L^{-1}[F(s)] = e^t$, $L^{-1}[G(s)] =$

$f(t)$

$= L^{-1}\left[\frac{1}{(s+2)^2}\right]$

$= e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$

$L^{-1}[G(s)] = te^{-2t}$

$g(t) = te^{-2t}$

By Convolution Theorem:-

$$L^{-1}[F(s)\cdot G(s)] = \int_0^t F(u)\cdot g(t-u)\,du$$

$$= \int_0^t e^{u}\cdot (t-u)\, e^{-2(t-u)}\,du$$

$$= \int_0^t e^{3u}(t-u)\, e^{-2t}\,du$$

$$= e^{-2t}\int_0^t (t-u)e^{3u}\,du$$

$$= e^{-2t}\left[\frac{e^{3t}-3t-1}{9}\right]$$

(7) $L^{-1}\left[\frac{1}{s^2-s-12}\right] = L^{-1}\left[\frac{1}{(s-4)}\cdot\frac{1}{(s+3)}\right]$

$F(s) = \frac{1}{s-4}$, $G(s) = \frac{1}{s+3}$

$L^{-1}[F(s)] = e^{4t}$ $L^{-1}(G(s)) = e^{-3t}$

By Convolution Theorem,

$$L^{-1}[F(s)\cdot G(s)] = \int_0^t F(u)\cdot g(t-u)\,du$$

$$= \int_0^t e^{4u}\cdot e^{-3(t-u)}\,du$$

$$= \int_0^t e^{7u}\cdot e^{-3t}\,du$$

$$= \frac{e^{-3t}}{7}\left(e^{7u}\right)_0^t$$

$$= \frac{e^{-3t}}{7}\left(e^{7t}-1\right)$$

$$= \frac{e^{4t}-e^{-3t}}{7}$$

Page No.
Date

Application of Inverse Laplace Transform:-

1- Solve $\frac{d^2y}{dt^2} + y = t$; $y(0) = 1$
$y'(0) = 2$

$$\frac{d^2y}{dt^2} + y = t$$

Apply Laplace transform on both sides.

$$L[y''] + L[y] = L[t]$$
$$s^2 Y - s\,y(0) - y'(0) + Y = \frac{1}{s^2}$$
$$s^2 Y - s(1) + 2 + Y = \frac{1}{s^2}$$

$$Y(s^2+1) = \frac{1}{s^2} + (s-2)$$

$$Y = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$$

By Partial Fraction.

$$Y = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$L[y] = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$y = L^{-1}\left[\frac{1}{s^2}\right] - L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{s}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{s^2+1}\right]$$

$y = t - \sin t + \cos t - 2\sin t$
$y = t + \cos t - 3\sin t$

~~2. $y''+9y=\dots$, $y(0)=9$, $y'(0)=\dots$~~

2. $y''-9y=0$ with $y(0)=4$ & $y'(0)=9$

Apply Laplace transform.

$$L[y'']-9L[y]=L[0]$$
$$s^2L[y]-sy(0)-y'(0)-9L[y]=0$$
$$s^2Y-4s-9-9Y=0$$
$$Y=\frac{4s+9}{s^2-9}$$
$$L[y]=\frac{4s}{s^2-9}+\frac{9}{s^2-9}$$
$$y=4L^{-1}\left[\frac{s}{s^2-9}\right]+9L^{-1}\left[\frac{1}{s^2-9}\right]$$
$$y=4\cosh(3t)+\frac{9}{3}\sinh(3t)$$
$$y=4\cosh(3t)+3\sinh(3t)$$

Simultaneous Ordinary D.E :-

$$\frac{dx}{dt}=2x-3y,\quad \frac{dy}{dt}=y-2x,\quad x(0)=8,\ y(0)=3$$

Apply Laplace Transform.

$$L\left[\frac{dx}{dt}\right]=L[2x-3y]$$
$$s\cdot L[x]-x(0)=2L[x]-3L[y]$$
$$s\cdot L[x]-8=2L[x]-3L[y]$$
$$sX-8=2X-3Y$$
$$X(s-2)+3Y=8 \quad \leftarrow (1)$$

Page No.
Date

Similarly,

$$\frac{dY}{dt} = Y - 2X$$

Apply Laplace Transform,

$$L\left[\frac{dY}{dt}\right] = L[y - 2x]$$

$$s \cdot L[Y] - Y(0) = L[Y] - 2L[x]$$

$$sY - 3 = Y - 2X$$

$$2X + (s-1)Y = 8 \quad \text{---}(2)$$

Apply Cramer's Rule,

$$X = \frac{D_X}{D}, \quad Y = \frac{D_Y}{D}$$

$$D = \begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix} = s^2 - 3s - 4$$

$$D_X = \begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix} = 8s - 17$$

$$D_Y = \begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix} = 3s - 22$$

$$X = \frac{8s-17}{s^2-3s-4}, \quad Y = \frac{3s-22}{s^2-3s-4}.$$

By Partial Fraction,

$$L[x] = X = \frac{5}{s+1} + \frac{3}{s-4}$$

$$x = L^{-1}\left[\frac{5}{s+1}\right] + L^{-1}\left[\frac{3}{s-4}\right]$$

$$x = 5e^{-t} + 3e^{4t}$$

$$Y = \frac{5}{s+1} - \frac{2}{s-4}$$

$$y = L^{-1}\left[\frac{5}{s+1}\right] - L^{-1}\left[\frac{2}{s-4}\right]$$

$$y = 5e^{-t} - 2e^{4t}.$$

Page No.
Date :

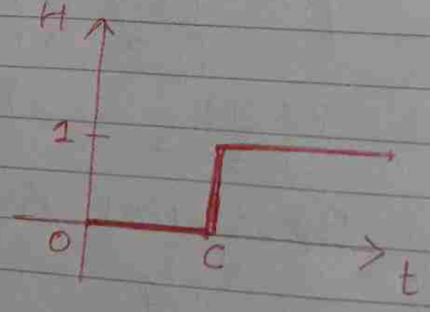
Laplace of Periodic function

Let f be a function of variable $t>0$
then f is said to be periodic
function of period T if
$f(t+T) = f(t)$, $T>0$
Ex:- $\sin\theta = \sin(2\pi+\theta)$
$\Rightarrow \sin\theta$ is periodic
with period 2π.

Let $f(t)$ be periodic with period $T>0$
then
$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-su} \cdot F(u)\,du.$$

Heaviside Unit step Function:-

$$H_c(t) = 0 \quad , 0 \le t < c$$
$$= 1 \quad , c \le t \quad \text{for } c \geq 0$$



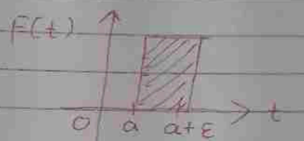
SY-NOTES

Page No.
Date

Dirac-Delta Function

Let F be a function of $t>0$ such that

$$F(t) = 0 \qquad 0<t<a$$
$$= 1/\varepsilon \qquad a \le t \le a+\varepsilon$$
$$= 0 \qquad a+\varepsilon < t$$



Now, $\int_0^\infty F(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1$

As $\varepsilon \to 0$, $F(t) \to \infty$ at $t=a$ & zero everywhere.

Dirac-Delta or Unit Impulse Function: $\delta(t-a) = \lim_{\varepsilon \to 0} F(t)$

If $a=0$ then

$$\delta(t) = \lim_{\varepsilon \to 0} F(t)$$