2.1 INTRODUCTION:

Every statement in mathematics must be precise and exact. The rules of logic specify the meaning of mathematical statements. In the previous chapter the basic conversions of logic was introduced. Here we have discussed, the conversion of a formal statement into symbolic language, using quantifiers such as for all or their exists. Every statement is based on proofs and each proof needs proper reasoning. There are many different types of proofs. In this chapter, we shall look at some more common type like direct proof, division into cases, proof by contraposition and proof by contradiction. Logic deals with the method of reasoning. It has practical applications to the design of computer machines, to computer programming, to computer languages, to artificial intelligence and to other fields of computer science. It is a base for electronics and electronics has open the gates for computer science so that we are all now into the era of Information Technology.

This chapter also consists of number theory. Different properties of rational and irrational numbers. We have discussed two classical theorems using indirect proof. These theorems were well known more than 2000 years ago.

In everyday life we come across with the concept of algorithms. In the situations like preparation of tea, sewing of cloths, filling some form etc. These involves sequence of logical steps, to be performed to achieve the aim.

Let us start with the logical statement.

2.2 PREDICATES:

We have seen in Illustration 1.4.1 (vii) that the sentence $x + 3 = 7$ is not a statement in logic since its truth value is depend upon the value of x.

$\therefore$ If $x = 4$, then $x + 3 = 7$ is true otherwise it is false.

Another sentence like $x > 7$, $x = y + 4$, $x + y = z$ are not statements in logic. Here we will discuss the ways that propositions can be produced from such statements.

Consider the sentence $x > 7$ i.e. x is greater than 7. This has two parts. One is the variable x and other is the property "is greater than 7". The property of this sentence is called predicate.

Predicate: The property of the sentence is called predicate which is denoted by P.

$\therefore$ $x > 7$ is denoted by P (x), where P is the predicate "is greater than 7" and x is the variable.

The sentence P (x) is also called propositional function P at x.

Illustration 2.2.1:

Let P (x) is $x > 7$. What are truth values of P (10) and P (3)?

Solution:

Here P (10) means the value at $x = 10$.

$\therefore$ $10 > 7$ is true $\therefore$ P (10) is true and since $3 > 7$ is false

$\therefore$ P (3) is false.

Illustration 2.2.2:

Let Q (x, y) is $x = y + 4$. What are the truth values of Q (– 3, 0), Q (2, – 2) and Q (6, 2)?

Solution:

$\therefore$ To get value Q (– 3, 0) put $x = -3$ & $y = 0$

$\therefore$ $-3 = 0 + 4$ i.e. $-3 = 4$ which is false.

$\therefore$ Q (–3, 0) is false.

Note that Q (x, y) is not a proposition but Q (3, 0) is a proposition.

For Q (2, – 2) put x = 2 & y = – 2

∴ 2 = – 2 + 4 ∴ 2 = 2 which is true

∴ Q (2, – 2) is true.

Also Q (6, 2) is true.

2.3 QUANTIFIERS:

We have seen when a particular value is assign to the variable let us say x in P (x) then propositional function becomes proposition since for that value of x it is either true or false. is one way to produce proposition from a sentence. However, there is another important w produce proposition from propositional function which is called quantification.

The words like all, some, many, none and few are used in quantification. Here we wil universal and existential quantification.

Universal Quantification of the proposition P(x) or Universal Statement: It is a statemen all values of x, P(x) is true. It is denoted by $\forall$ x P(x). The symbol $\forall$ is called universal quantifi

Illustration 2.3.1:

(i) Let P(x): – (– x) = x is a proposition is true for all value of x where x is a real number.

∴ The universal quantification of P(x); $\forall$ x P(x) is a true statement.

(ii) Let Q(y): $y^2 = -4$.Then $\forall$ y Q(y) is a false statement for any real number y.

Existential quantification of the proposition P(x): It is a statement in which P(x) is tru some values of x i.e. There exist a value of x for which P(x) is true. The existential quantific of P(x) is denoted by $\exists$ x P(x). The symbol $\exists$ is called existential quantifier.

Let P(y): y + 1 < 4 then $\exists$ y. s. t. P(y) is true.

e.g. y = 1, 2, – 5, ... etc. But for y = 5, 10, ... etc. the statement is false.

Illustration 2.3.2:

(i): P(x) x is even

∴ The existential quantification of P(x); $\exists$x P(x) is a true statement.

(ii) The statement $\exists$y y + 2 = y is a false. There is no value of y for which proposition P(2 = y produces a true statement.

Illustration 2.3.3:

Rewrite the following statements using variables and quantifiers:

(i) All quadrilaterals have four sides.

(ii) Sum of all angles of a triangle is 180°.

(iii) No snakes have hands.

(iv) Some numbers are perfect numbers.

Solution:

Let Q, T, S, N are sets of all quadrilaterals, triangles, snakes, numbers respectively.

(i) $\forall$ q $\in$ Q, q has four sides.

(ii) $\forall$t $\in$ T, sum of three angles = 180°.

(iii) $\forall$ s $\in$ S, s does not have hands.

(iv) $\exists$ x $\in$ N such that x is perfect.

Illustration 2.3.4:

Let B ={–56, –39, –5, 0, 8, 12, 50, 56}. Determine which of the following statements are true or false:

(i) $\forall x \in B$, if x is even then $x > 0$

(ii) $\exists x \in B$, such that x and –x both are in B

(iii) $\forall x \in B$, if x is odd then $x < 0$

(iv) $\forall x \in B$, if x is non zero even number then it is divisible by 4

(v) $\forall x \in B$, if the ones digit of x is 6, then tens digit is –5 or 5.

Solution:

(i) False $\because$ –56 is even but $-56 \not> 0$

(ii) True $\because$ 56, $-56 \in B$

(iii) True $\because$ –39, –5 are odd < 0

(iv) False $\because$ 50 is non zero even number, not divisible by 4

(v) True $\because$ –56, $56 \in B$

2.4 RULES FOR NEGATION:

Negation of the Statement: Negation of the given simple statement P is ~P.

Rules for writing Negation of a symbolic statement:

(1) If P is a compound statement which involves only ~, $\vee$ and $\wedge$ then the negation of P denoted by ~ P and is obtained by interchanging every $\vee$ with $\wedge$ and vice-a-versa and replacing every statement by its negation.

(2) If the statement involve $\rightarrow$ or $\leftrightarrow$ then we reunite the statement in such a way that it contains only ~, $\vee$ or $\wedge$ and then write its negation by rule (1).

Illustration 2.4.1:

Using the rules for negation, write down negation for the following symbolic statements.

(i) $(p \vee q) \wedge r$ (ii) $(p \wedge q) \vee (q \vee \sim r)$ (iii) $(p \vee q) \rightarrow r$ (iv) $p \rightarrow (q \wedge r)$

Solution:

(i) $\sim [(p \vee q) \wedge r] \sim (p \vee q) \vee \sim r$

$(\sim p \wedge \sim q) \vee \sim r$

(ii) $\sim [(p \wedge q) \vee (q \vee \sim r)]$

$\sim (p \wedge q) \wedge \sim (q \vee \sim r)$

$(\sim p \vee \sim q) \wedge (\sim q \wedge r)$

(iii) $\sim [(p \vee q) \rightarrow r]$

$\sim [\sim (p \vee q) \vee r]$

$(p \vee q) \wedge \sim r$

(iv) $\sim [p \rightarrow (q \wedge r)]$

$\sim [\sim p \vee (q \wedge r)]$

$p \wedge \sim (q \wedge r)$

$p \wedge (\sim q \vee \sim r)$

Rules for writing Negation of a Verbal Statement:

(1) If the given compound statement involving, OR and AND then its negation is obtain interchanging every OR by AND and vice a versa.

(2) If the given compound statement involves NOT then negation is obtained by deletin word 'NOT'.

(3) If the given compound statement does not involve 'NOT' then negation is obtain inserting the word 'NOT'.

(4) If P(x) is true for universal quantifier then replace universal quantifier by exist quantifier and vice a versa.

(5) If the given statement involves the word "EVERY" or "ALL" then its negation is obt by replacing it by "SOME or THERE EXIST ATLEAST ONE" and vice versa.

Illustration 2.4.2:

Write the negation of the following statements:

(i) Some girls are sincere.

(ii) I will have tea or coffee.

(iii) You will be smart if and only if you are healthy.

(iv) All men are animals.

(v) If I am not in a good mood, then I will go to a movie.

(vi) The weather is bad and I will not go to work.

Solution:

Negations of the above statements are as follows:

(i) Every girl is not sincere.

(ii) I will not have tea and coffee.

(iii) You will be smart but you will not be healthy or you will be healthy but you will smart.

(iv) There exist a man who is not an animal.

(v) I am not in a good mood but I will not go to a movie.

(vi) The weather is not bad or I will go to work.

Negation of quantified statements:

Consider the statement "All persons in this group are vegetarian".

Its negation is "It is not true that all persons in this group are vegetarian" i.e. there is one person who is non-vegetarian. Equivalently we can say that, "there exists at least one who is not vegetarian (or who is non-vegetarian).

To write this statement symbolically, let G denote the group of persons, then the statement can be written as

$\sim(\forall\ x \in G)$ (x is vegetarian) $\equiv (\exists\ x \in G)$ (x is non-vegetarian) or when P(x) denote vegetarian".

We have

$\sim(\forall\ x \in G)\ P(x) \equiv (\exists\ x \in G) \sim P(x)$ or $\sim\forall\ x\ P(x) \equiv \exists\ x \sim P(x)$

Negation of a universal statement $\forall$ is logically equivalent to an existential $\exists$ statement.

Consider another statement.

"There is a person in this group who is vegetarian". Symbolically this statement is writ

x P(x)

Its negation is "It is not true that there is a person in this group who is vegetarian". Equivalently "All persons in this group are not vegetarian".

Symbolically $\forall x \sim P(x)$

$\therefore \sim\exists x\, P(x) \equiv \forall x \sim P(x)$

These rules for negation of quantifiers are called **DeMorgans laws** for **quantifiers**. Following table illustrates negation of quantified statements.

Statement		Negation	
$\forall x\, P(x)$	(all true)	$\exists x\, (\sim P(x))$	(at least one false)
$\exists x\, (\sim P(x))$	(at least one false)	$\forall x\, P(x)$	(all true)
$\forall x (\sim P(x))$	(all false)	$\exists x\, P(x)$	(at least one true)
$\exists x\, P(x)$	(at least one true)	$\forall x (\sim P(x))$	(all false)

Illustration 2.4.3:

Negate each of the following statements: For $G = \{1, 2, 3, 4, 5\}$

(a) $(\exists x \in G)\,(x + 3 = 10)$

(b) $(\forall x \in G)\,(x + 3 < 10)$

(c) $(\exists x \in G)\,(x + 3 < 5)$

(d) $(\forall x \in G)\,(x + 3 \leq 7)$

Solution:

(a) $(\forall x \in G)\,(x + 3 \neq 10)$

(b) $(\exists x \in G)\,(x + 3 \geq 10)$

(c) $(\forall x \in G)\,(x + 3 \geq 5)$

(d) $(\exists x \in G)\,(x + 3 > 7)$

Illustration 2.4.4:

Negate each of the following statements:

(a) All the voters are 18 and above.

(b) For all real numbers x, if $x > 5$ then $x^2 > 25$.

(c) There is an honest shopkeeper.

(d) Square of all non zero real numbers are positive.

Solution:

(a) Some voters are under 18.

(b) Let P(x) is $x > 5$ and Q(x) is $x^2 > 25$.

The given statement is $\forall x\, (P(x) \rightarrow Q(x))$ which is equivalent to $\forall x\, (\sim P(x)) \vee Q(x))$.

$\therefore$ The negation is $\exists x\, (P(x) \wedge \sim Q(x))$ i.e. there exists a real number x such that $x > 5$ and $x^2 \leq$ 5.

(c) Let S is the set of shopkeepers, P(x) is "x is honest". The given statement is $(\exists x \in S)\, P(x)$ or x P(x). The negation is $\forall x (\sim P(x))$ i.e. All shopkeepers are not honest.

(d) The statement is $\forall x \in \mathbb{R} - \{0\},\ x^2 > 0$

The negative is $\exists x \in \mathbb{R} - \{0\},\ x^2 \leq 0$

5 MULTIPLE QUANTIFIERS:

These are the predicates which are functions of more than one variable. The domains of these ariables may or may not be same. In such cases multiple quantifiers are used.

There are eight ways to apply the two quantifier $\forall$ and $\exists$ for two variables say x and y.

e.g., Consider the statement "x is related to y".

∴ p(x, y) : x is relative of y. Then

(1) $(\forall x)(\forall y) P(x, y)$: Everyone is related to everybody

(2) $(\forall x)(\exists y) P(x, y)$: Everyone is related to someone.

(3) $(\exists x)(\forall y) P(x, y)$: Someone is related to everyone.

(4) $(\exists x)(\exists y) P(x, y)$: Someone is related to somebody.

(5) $(\forall y)(\forall x) P(x, y)$: Everybody is related by everyone.

(6) $(\forall y)(\exists x) P(x, y)$: Everybody is related by someone.

(7) $(\exists y)(\forall x) P(x, y)$: There is someone whom everyone is related.

(8) $(\exists y)(\exists x) P(x, y)$: There is someone whom someone is related.

Here the order of the quantifiers is very important.

∴ $\forall x \exists y P(x, y)$ is not same as $\exists y \forall x P(x, y)$.

But $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$ and

$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$

Illustration 2.5.1:

(a) Let the statement is P(x, y): For every y, there exist x such that $x + y = 5$.

The given statement is written as $\forall y \exists x, x + y = 5$.

This statement is true if $y = 2$, then $x = 3$, if $y = 1$, then $x = 4$ and so on i.e. if we conside y, then $x + y = 5$ is true for all y. By reversing order of quantifiers we get $\exists x \forall y, x + y = 5$

i.e. there exists a single value of x such that for every y, $x + y = 5$ which is not true. Hen statement is false.

(b) $\forall x \in Z^+ \exists y \in Z^+$ such that $x = y + 1$. This statement is false ∵ $x = 1 \in Z^+$ but $y = 0 \notin$

∴ For $x = 1$ there is no $y \in Z^+$ such that $x = y + 1$

Illustration 2.5.2:

Negate each of the following statements.

(a) $\exists x \forall y, P(x, y)$ **(b)** $\forall x \forall y, P(x, y)$ **(c)** $\exists y \exists x \forall z, P(x, y, z)$

(d) $\forall x \exists y \exists z, P(x, y, z)$ **(e)** $\forall x \in A (\forall y \in B (P(x, y)))$

Solution:

Use $\sim \forall x P(x) \equiv \exists x \sim P(x)$

And $\sim \exists x P(x) \equiv \forall x \sim P(x)$

(a) $\sim (\exists x \forall y, P(x, y)) \equiv \forall x \exists y, \sim P(x, y)$

(b) $\sim (\forall x \forall y, P(x, y)) \equiv \exists x \exists y \sim P(x, y)$

(c) $\sim (\exists y \exists x \forall z, P(x, y, z)) \equiv \forall y \forall x \exists z, \sim P(x, y, z)$

(d) $\sim (\forall x \exists y \exists z, P(x, y, z)) \equiv \exists x \forall y \forall z, \sim P(x, y, z)$

(e) $\sim (\forall x \in A (\forall y \in B (P(x, y)))) \equiv \exists x \in \forall (\exists y \in B (\sim P(x, y)))$

EXERCISE 2.1

(1) Write down the negations for each of the following:

(i) $(\sim p \wedge q) \vee \sim r$ **(ii)** $p \leftrightarrow \sim q$ **(iii)** $p \rightarrow (q \rightarrow r)$ **(iv)** $\forall x \in C$, C has CPU, where C is set of compu

(2) Let A = {1, 2, 3, 4, 5}. Determine the truth value of each of the following statements.

(a) Let P(x): $x + 3 = 10, \exists x \in A, P(x)$ **(b)** Let P(x): $x + 3 < 10, \forall x \in A, P(x)$

(c) Let P (x): $x + 3 < 5, \exists x \in A, P(x)$ (d) Let P (x): $x + 3 \leq 7, \forall x \in A, P(x)$

(3) Negate the following statements:

(i) For all girls g, if g is simple then g has long hair.

(ii) For all natural numbers n, if n is divisible by 6, then n is divisible by 3.

ANSWERS 2.1

(1) (i) $(p \vee \sim q) \wedge r$ (ii) $(p \wedge q) \vee (\sim q \wedge \sim p)$ (iii) $p \wedge (q \wedge \sim r)$ (iv) $\exists x \in C$, C has not a CPU.

(2) (a) False $\because x + 3 = 10 \Rightarrow x = 7 \notin A$ (b) True, $\because$ any $x \in A$ satisfy $x + 3 < 10$

(c) True $\because 1 + 3 < 5$ & $1 \in A$ (d) False, $\because$ for $5 \in A$ but $5 + 3 \nless 7$

(3) (i) $\exists$ g such that g is simple and g has not long hair.

(ii) $\exists$ n such that n is divisible by 6 and n is not divisible by 3.

(c) $q \rightarrow p$: If the number is divisible by 2 then it is an even integer.

(d) $\sim p \rightarrow \sim q$: If an integer is not even then it is not divisible by 2.

(e) $\sim q \rightarrow \sim p$: If the number is not divisible by 2 then it is not an even integer.

2.6 ARGUMENT WITH QUANTIFIED STATEMENTS:

In this section we will study the use of quantifiers to express different types of statements.

The rule of Universal instantiation:

If some property is true for every object in a set, then it is true for any particular object in a set.

Consider the example of universal instantiation.

Illustration 2.6.1:

All men are wise.

Jay is a man

$\therefore$ Jay is wise

Convert this argument using quantifiers.

Solution:

Let P(x):x is man

Q(x): x is wise

Note that first two statements are premises and the third is conclusion.

$\therefore$ We can express the given argument as follows:

$\forall x (P(x) \rightarrow Q(x))$

P(a), where a is a particular object in the set of men, here a = Jay.

$\therefore$ Q(a) i.e. Jay is wise.

Illustration 2.6.2:

All tigers are violent.

Some tigers do not eat grass.

$\therefore$ Some violent animals do not eat grass.

Convert this argument using quantifiers.

Solution:

Let P(x): x is a tiger

Q(x): x is violent.

R(x): x eats grass.

The given argument is:

$\forall x (P(x) \rightarrow Q(x))$

$\exists x (P(x) \wedge \sim R(x))$

$\exists x (Q(x) \wedge \sim R(x))$

Illustration 2.6.3:

Convert the following arguments using quantifiers and also check its validity using diagra

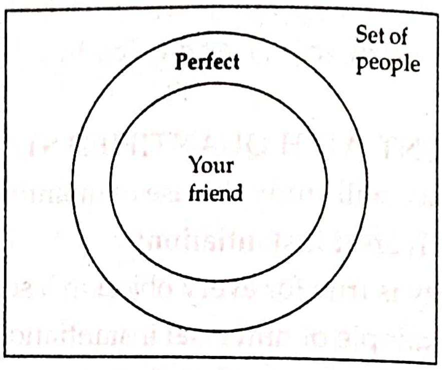
(a) All your friends are perfect. (b) Not everyone is perfect.

Solution:

Let P(x): x is perfect

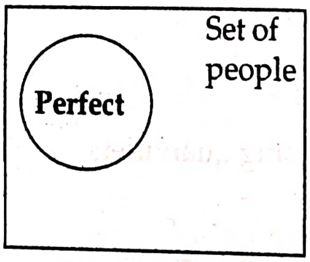
Q(x): x is your friend.

The domain is all people.



(a) $\forall x (Q(x) \rightarrow P(x))$

Fig. 2.6.1



(b) $\sim \forall x P(x) \equiv \exists x \sim P(x)$

Fig. 2.6.2

EXERCISE 2.2

Convert the following arguments using quantifiers. State its validity:

(1) All professors are dreamy.
Ketan is not dreamy.
∴ Ketan is not professor.

(2) All late comers sit in the back row.
Mahesh sits in the back row.
∴ Mahesh is late comer.

(3) All human beings are honest.
Rohini is honest.
∴ Rohini is human being.

(4) No cat eat grass.
Mani is a cat.
∴ Mani does not eat grass.

ANSWERS 2.2

(1) P(x): x is professor

Q(x): x is dreamy

Then the argument is

$\forall x, P(x) \rightarrow Q(x)$

$\sim Q(K)$ K for Ketan

$\therefore \sim P(K)$

$\therefore$ Valid

(2) P(x): x is late comer

Q(x): x sits in back row

Then the argument is

$\forall x, P(x) \rightarrow Q(x)$

Q(M) M for Mahesh

$\therefore P(M)$

$\therefore$ Valid

(3) P(x): x is human being

Q(x): x is honest

Then the argument is

$\forall x, P(x) \rightarrow Q(x)$

Q(R) R for Rohini

$\therefore P(R)$ or $\sim P(R)$

$\therefore$ Invalid

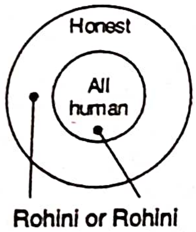
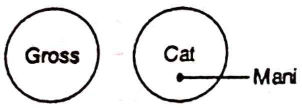


Fig. AE - 1

(4)



$\therefore$ Valid Fig. AE - 2

.7 METHODS OF PROOFS:

Till now we have seen the mathematical proofs which are based on some logical reasoning. Ve start with the given fact/s and using logical reasoning deduce other facts. Continuing in this vay, till we reach to the aim. Let us see the method of direct proof.

irect Proof:

It is the most familiar type of proof. The method to prove the conditional statement $p \rightarrow q$ is ue, is to assume the original statement p is true and use this fact to show directly that the nother statement q is true.

Following steps can be used for direct proof:

- Start with given fact i.e. here assume p is true.
- Use other facts about statement p to conclude that statement q is true, this will s $p \rightarrow q$ is true.

Here p is the hypothesis and q is the conclusion. p may be compound statement like H_1 $\wedge \ldots \wedge H_n$ or $H_1 \vee H_2 \vee \ldots \vee H_n$.

Let us solve some examples on direct proofs.

Illustration 2.7.1:

Using direct proof, prove that addition of two rational numbers is a rational number.

Solution:

Remember that a rational number is a number which can be written in the form of $\frac{m}{n}$, r m, n $\in \mathbb{Z}$.

Let x and y are two rational numbers then $\exists$ m, n, r, s $\in \mathbb{Z}$ such that $x = \frac{m}{n}$ and $y = \frac{r}{s}$ wher 0, $s \neq 0$.

Consider $x + y = \frac{m}{n} + \frac{r}{s} = \frac{ms + rn}{ns}$

since $n \neq 0, s \neq 0$, $\therefore$ $ns \neq 0$ and also m, n, r, s $\in \mathbb{Z}$, $\therefore$ ms + rn, ns $\in \mathbb{Z}$.

$\therefore$ x + y is a rational number.

$\therefore$ Addition of two rational numbers is a rational number.

Illustration 2.7.2:

Using direct proof, prove that product of two odd integers is an odd integer.

Solution:

Let x and y are odd integers. Then $\exists$ r, s $\in \mathbb{Z}$ such that

$x = 2r + 1$ and $y = 2s + 1$ — By definition of odd number

Consider $xy = (2r + 1)(2s + 1)$

$= 4rs + 2r + 2s + 1$ — By expanding the brackets

$= 2(2rs + r + s) + 1$ — Taking 2 common

$\because r, s \in \mathbb{Z} \Rightarrow 2rs + r + s \in \mathbb{Z}$

Let $l = 2rs + r + s$

$= 2l + 1$ which is odd. — By definition of odd number

$\therefore$ Product of two odd integers is an odd integer.

Illustration 2.7.3:

Using direct proof, prove that the sum of one even integer and an odd integer is an integer.

Solution:

Let x is an even integer and y be an odd integer.

Then $\exists$ r, s $\in \mathbb{Z}$ such that

$x = 2r$ and $y = 2s + 1$ — By definition of odd and even number

Consider $x + y = 2r + (2s + 1)$

$= 2(r + s) + 1$ Taking 2 common

$= 2l + 1$ $\because r, s \in \mathbb{Z} \Rightarrow l = 2(r + s) \in \mathbb{Z}$

which is odd.

Illustration 2.7.4:

Using direct proof, prove that if n is an even integer then 7n + 4 is an even integer.

Solution:

Let $n = 2r$ for some $r \in \mathbb{Z}$

Consider $7n + 4 = 7(2r) + 4$ Put value of n

$= 14r + 4$ Expand bracket

$= 2(7r + 2)$ Taking 2 common

$= 2l$ Let $l = 7r + 2 \in \mathbb{Z} \because r \in \mathbb{Z}$

$\therefore$ 7n + 4 is an even integer.

Illustration 2.7.5:

Prove that square of odd integer is odd.

Solution:

Let x is an odd integer.

$\therefore x = 2r + 1$ for some $r \in \mathbb{Z}$

$\therefore x^2 = (2r + 1)^2$

$= (2r + 1)(2r + 1)$

$= 4r^2 + 2r + 2r + 1$ Expanding Brackets

$= 4r^2 + 4r + 1$ Combining similar terms

$= 2(2r^2 + 2r) + 1$ Taking 2 common

$= 2l + 1$ $\because r \in \mathbb{Z} \Rightarrow 2r^2 + 2r = l \in \mathbb{Z}$

$\therefore x^2$ is odd.

In mathematical logic the given statement is written as $\forall x\, p(x) \rightarrow q(x)$ where

p(x):x is an odd integer

q(x):x^2 is odd

So we have started with the assumption that the hypothesis of this conditional statement is true i.e. p(x) is true and using the definition of odd integer we have proved that q(x) is also true.

Illustration 2.7.6:

Prove that $1 + 2 + 3 + \ldots + n = \dfrac{n(n + 1)}{2}$

Solution:

Let $X = 1 + 2 + 3 + \ldots + n$ [LHS of given statement]

Then $X = n + (n - 1) + (n - 2) + \ldots 3 + 2 + 1$ [By commutative law]

$\therefore X + X = (n + 1) + (n - 1 + 2) + (n - 2 + 3) + \ldots + (1 + n)$ [By term by term addition]

$\therefore 2X = (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1)$ [n times]

$\therefore 2X = n(n + 1)$

$\therefore X = \frac{n(n+1)}{2}$

$\therefore 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Illustration 2.7.7:

Prove that $\forall$ x, y $\in \mathbb{Z}^+$, if x divides y, then $x \leq y$.

Solution:

Given x divides y, $\therefore \exists$ k $\in \mathbb{Z}^+$ such that y = xk $\because x, y \in \mathbb{Z}^+$

As $k \in \mathbb{Z}^+ \Rightarrow k \geq 1$ i.e. $1 \leq k$

$\Rightarrow x \leq xk$ Multiplying both sides by $x \in \mathbb{Z}^+$

$\Rightarrow x \leq y$ $\because y = xk$

2.8 DIVISION IN THE INTEGERS

We will now discuss some important results about division and factoring in the integers which are needed later.

Theorem 2.8.1:

If n and m are integers and n > 0 we can write m = qn + r for integers q and r with $0 \leq r < n$

Illustration 2.8.1:

Write m as qn + r with $0 \leq r < n$

(i) m = 20, n = 3

m = qn + r

20 = 6 (3) + 2

$\therefore$ r = 2

(ii) m = 64, n = 37

m = qn + r

64 = 1 (37) + 27

$\therefore$ r = 27

If r is zero then we say that m is multiple of n. We write n | m which we read "n divides m". i.e. If n | m then m = qn and $n \leq m$. Here q is quotient after dividing m by n. If m is not a multiple of n, we write $n \nmid m$ which we read as "n does not divide m". Note that n/m denote the number n is divided by m.

Properties of Divisibility:

Theorem 2.8.2:

Let a, b and c be integers.

(i) If a | b and a | c then a | b + c

(ii) If a | b and a | c where b > c, then a | b − c

(iii) If a | b or a | c then a | bc

(iv) If a | b and b | c then a | c

Proof:

(i) If a | b and a | c then for some integers k_1 and k_2,

$b = k_1 a$ and $c = k_2 a$

$\therefore \quad b + c = (k_1 + k_2)\, a$

$\therefore \quad a \mid (b + c)$

(ii) If $a \mid b$ and $a \mid c$ then for some integers k_1 and k_2,

$b = k_1 a$, and $c = k_2 a$

$\because \quad b > c \quad \therefore\ b - c > 0$

$\therefore \quad b - c = (k_1 - k_2)\, a$

$\therefore \quad a \mid (b - c)$

(iii) We have $b = k_1 a$ and $c = k_2 a$ where k_1 and k_2 are some integers.

$\therefore \quad bc = k_1 ac$ or $bc = k_2 ab$

In either case bc is a multiple of a

$\therefore \quad a \mid bc$

(iv) We have $b = k_1 a$ and $c = k_2 b$

$\therefore \quad c = k_2 b = k_2 (k_1 a)$

$\therefore \quad c = (k_1 k_2)\, a$

$\therefore \quad a \mid c$

Note that: If $a \mid b$ and $a \mid c$ then $a \mid (mb + nc)$ where m and n are any integers.

Algorithm for testing whether the number is prime or not

ALGORITHM to test whether an integer $n > 1$ is prime.

Steps : (1) Check whether n is 2 if so n is prime if not proceed to next step.

(2) Check whether $2 \mid n$ if so n is not prime otherwise proceed to next step.

(3) Compute the largest integer $k \le \sqrt{n}$ then

(4) Check whether $d \mid n$ where d is any odd number such that $1 < d \le k$. If $d \mid n$ then n is not prime otherwise n is prime.

Illustration 2.8.2:

Prove that if a and b are positive integers such that $a \mid b$ and $b \mid a$, then $a = b$.

Solution:

$\because a > 0$ and $a \mid b$

$\therefore$ there exist some integer k_1 such that $b = k_1 a$ ... (1)

Similarly,

$\because b > 0$ and $b \mid a$

$\therefore$ there exist some integer k_2 such that $a = k_2 b$ ... (2)

From (1) and (2)

$\because \quad b = k_1 a$

$b = k_1 (k_2 b) = k_1 k_2 b$

$\because \quad b \neq 0$

$\because \quad k_1 k_2 = 1$

$\because$ k_1 and k_2 are +ve integers

$\therefore$ $k_1 = k_2 = 1$

$\therefore$ from (2)

$$a = k_2 b$$

$\because$ $k_2 = 1$

$\therefore$ $a = b$

EXERCISE 2.3

Prove directly that:

(1) Product of one even and one odd integer is an even integer.

(2) If $x, y \in \mathbb{Z}$ are perfect squares then xy is also a perfect square.

(3) The difference of any two consecutive integers is odd.

(4) The sum of two even integers is even.

(5) Prove that $a + b$ and $b + c$ are even integers, where $a, b, c \in \mathbb{Z}$, then $a + c$ is an even integer.

(6) If $n \in \mathbb{Z}^+$, then n is odd if and only if $5n + 6$ is odd.

(7) The sum of three odd numbers is an odd number.

(8) If a and b are both square numbers then their product is also a square number.

(9) $\forall a, b \in \mathbb{Z}$, $(a + b)$ and $(a - b)$ are either both odd or both even.

(10) If x and y are rational numbers then $x^2 + y$ is also a rational number.

2.9 INDIRECT ARGUMENTS: CONTRADICTION AND CONTRAPOSITION:

Indirect Proof:

The proof which are not direct are called indirect proof. These proofs do not start with hypothesis and end with the conclusion. But they use negation of the conclusion.

There are two types of indirect proofs.

(i) Contrapositive proof or contraposition proof and

(ii) Proof by contradiction.

(i) Contrapositive Proof or Contraposition proof:

This proof uses the fact that the conditional statement $p \rightarrow q$ is equivalent t contrapositive, $\sim q \rightarrow \sim p$.

$\therefore$ To prove $p \rightarrow q$ we will prove $\sim q \rightarrow \sim p$. Here we take $\sim q$ as a hypothesis and related facts, we will show it will follow $\sim p$.

When easily we are not getting a direct proof, then proof by contraposition is very us

(ii) Proof by contradiction:

In this type of proof, we assume the opposite of what we have to prove, and by lo reasoning we get a contradiction. Hence we can conclude that our assumption was wr Therefore the original statement is true i.e. if we want to prove that the statement p is then we use contradiction that, assume $\sim p$ is true and prove that $\sim p \rightarrow q$ is true bu statement q is false (we know that, if $\sim p$ is T, q is F then $\sim p \rightarrow q$ must be false).

$\therefore$ We can conclude $\sim p$ is false $\rightarrow$ p is true.

Let us solve some examples on both the types of indirect proofs.

Illustration 2.9.1:

If $n \in \mathbb{Z}$ and $n^3 + 5$ is odd, then n is even.

Solution:

(i) Proof by contraposition:

Here $p: n^3 + 5$ is odd (Original hypothesis)

q: n is even (Original conclusion)

The contrapositive of the given conditional statement $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

$\therefore \sim q$: n is odd

$\sim p : n^3 + 5$ is even

Now we will take $\sim q$ as hypothesis and prove that $\sim p$ is true.

Let n is odd $\Rightarrow n = 2k + 1$ for some $k \in \mathbb{Z}$

$\therefore n^3 + 5 = (2k + 1)^3 + 5$ Substitute $n = 2k + 1$

$= 8k^3 + 12k^2 + 6k + 1 + 5$

$= 2(4k^3 + 6k^2 + 3k + 3)$

$= 2K$ where $K = 4k^3 + 6k^2 + 3k + 3 \in \mathbb{Z}$

i.e. $n^3 + 5$ is even.

$\therefore \sim p$ is true i.e. p is false.

$\sim p$ is the negation of the original hypothesis i.e. $\sim q$ implies p is false (But p is true given)

$\therefore$ The original conditional statement is true.

(ii) Proof by contradiction:

Here we have to prove that if $n^3 + 5$ is odd then n is even, so we will assume the contradiction that n is odd.

Let us assume that n is odd.

$\therefore n = 2k + 1$ for some $k \in \mathbb{Z}$

$n^3 + 5 = (2k + 1)^3 + 5 = 2K$ which is even as above

$\therefore n^3 + 5$ is even but it is given that $n^3 + 5$ is odd.

$\therefore$ Our assumption that n is odd is wrong

$\therefore$ n is even,

Observe that the steps followed in both proofs are same.

Illustration 2.9.2:

Prove that $\sqrt{3}$ is irrational. Use proof by contradiction.

Solution:

Let us assume that $\sqrt{3}$ is rational.

$\therefore \sqrt{3}$ rational $\Rightarrow \exists\, a, b \in \mathbb{Z}$ such that $\sqrt{3} = \frac{a}{b}$, $b \neq 0$, a and b have no common factors.

$\therefore \sqrt{3} = \frac{a}{b}$ By definition of rational numbers

$\therefore 3 = \frac{a^2}{b^2}$ Squaring both sides

$3b^2 = a^2 \Rightarrow a^2$ is multiple of 3

$\Rightarrow$ a is multiple of 3

[Otherwise $a \notin \mathbb{Z}$ e.g. $a^2 = 12$ then a^2 is multiple of 3 but $a = \sqrt{12} \notin \mathbb{Z}$ ∴ a must be multiple

Since a is a multiple of 3 it can be written as $a = 3k$ for some $k \in \mathbb{Z}$

∴ $3b^2 = a^2$ gives $3b^2 = (3k)^2$ ∴ $3b^2 = 9k^2$

∴ $b^2 = 3k^2$ $\Rightarrow b^2$ is a multiple of 3

$\Rightarrow$ b is also a multiple of 3

i.e. a and b both have a common factor 3.

But $\sqrt{3} = \frac{a}{b}$ are such that a and b have no common factor.

∴ Our assumption $\sqrt{3}$ is rational is wrong.

∴ $\sqrt{3}$ is irrational.

[Note that let $n = \frac{45}{30} = \frac{15 \times 3}{15 \times 2} = \frac{3}{2} = \frac{a}{b}$ so we can write rational number as $\frac{a}{b}$ with no com factor between a and b].

Relation between proof by contraposition and proof by contradiction:

The proof by contraposition can be converted into proof by contradiction. Using the sequence of steps used in proof by contraposition.

We have to prove that if p then q.

Proof by Contraposition	Proof by Contradiction
Take contrapositive i.e. $\sim q \rightarrow \sim p$	p is given to be true.
Assume $\sim q$ as hypothesis	Assume $\sim q$ is true
Step 1:	Step 1:
2:	2:
⋮	⋮
n:	n:
Conclude $\sim p$ is true	Conclude $\sim p$ is true
∴ If $\sim q$ then $\sim p$	Contradiction: p and $\sim p$ both are true
implies if p then q	∴ q is true

EXERCISE 2.4

Use contraposition proof and contradiction proof for example 1 to 4

(1) Prove that, if $n \in \mathbb{Z}$ and $3n + 2$ is even then n is even.

(2) Prove that, if $n \in \mathbb{Z}$ and n^2 is odd, then n is odd.

(3) Prove that, if $n \in \mathbb{Z}$ and $3n + 2$ is odd, then n is odd.

(4) Prove that if $n = ab$, a, b are positive, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

(5) Prove by contradiction, for any $x \in \mathbb{Z}$, $x^2 - 2$ is not divisible by 4.

(6) Prove by contradiction $\sqrt{5}$ is irrational.

2.10 DIVISION INTO CASES:

The proof by cases method make use of the equivalence $(H_1 \vee H_2 \vee \ldots \vee H_n) \rightarrow q$ is true by showing that $(H_1 \rightarrow q) \wedge (H_2 \rightarrow q) \wedge (H_3 \rightarrow q) \wedge \ldots \wedge (H_n \rightarrow q)$ is true.

Following are the steps:

(1) Divide the proposition into all possible cases.

(2) List all the cases.

(3) For each case, prove that the proposition is true.

(4) Conclude that the given proposition is true.

Illustration 2.10.1:

Prove that $\forall\, n \in \mathbb{Z}^+$, $n^3 + n$ is even.

Solution:

For a positive integer n, there are two possibilities:

(i) n is even (ii) n is odd.

We have to prove the given conclusion i.e. $n^3 + n$ is even is true for both cases.

Case (i): n is even

$\Rightarrow n = 2k$ for some $k \in \mathbb{Z}^+$

$\therefore n^3 + n = (2k)^3 + 2k$ — Substitute n = 2k

$= 8k^3 + 2k$

$= 2(4k^3 + k)$ — Take 2 common

$= 2K$ is even — By definition of even number $\because K = 4k^3 + k \in \mathbb{Z}^+$

$\therefore$ n is even $\Rightarrow n^3 + n$ is even

Case (ii): n is odd

$\Rightarrow n = 2k + 1$ for some $k \in \mathbb{Z}^+$

$\therefore n^3 + n = (2k + 1)^3 + (2k + 1)$

$= 8k^3 + 12k^2 + 6k + 1 + 2k + 1$

$= 8k^3 + 12k^2 + 8k + 2$

$= 2(4k^3 + 6k^2 + 4k + 1)$ — $\because K = 4k^3 + 6k^2 + 4k + 1 \in \mathbb{Z}^+$

$= 2K$ is even

$\therefore$ n is odd $\Rightarrow n^3 + n$ is even

In all cases the conclusion is true.

$\therefore \forall\, n \in \mathbb{Z}^+$, $n^3 + n$ is even.

llustration 2.10.2:

Prove that $|xy| = |x|\,|y|$

olution:

Divide into following cases:

Case (i):

$x \geq 0, y \geq 0 \therefore xy \geq 0 \Rightarrow |xy| = xy$ and $|x| = x$, $|y| = y$

$\therefore |xy| = xy = |x|\,|y|$

Case (ii):

$x \geq 0, y < 0 \therefore xy \leq 0 \Rightarrow |xy| = -xy$, and $|x| = x, |y| = -y$

$\therefore |xy| = -xy = x(-y) = |x|\ |y|$

Case (iii):

$x < 0, y < 0 \therefore xy \geq 0 \Rightarrow |xy| = xy$, and $|x| = -x, |y| = -y$

$\therefore |xy| = xy = (-x)(-y) = |x|\ |y|$

Case (iv):

$x < 0, y \geq 0 \therefore xy \leq 0 \Rightarrow |xy| = -xy$ and $|x| = -x, |y| = y$

$\therefore |xy| = -xy = (-x)(y) = |x|\ |y|$

$\therefore |xy| = |x|\ |y|$ is true in all cases.

$\therefore |xy| = |x|\ |y|$

Illustration 2.10.3:

Prove that if $x \in \mathbb{R}$ then $|x + 3| - x > 2$

Solution:

Case (i) Assume $x \geq -3 \Rightarrow x + 3 \geq 0 \Rightarrow |x + 3| = x + 3$

$\therefore |x + 3| - x = x + 3 - x = 3 > 2$

$\therefore |x + 3| - x > 2$

Case (ii) Assume $x < -3 \Rightarrow x + 3 < 0 \Rightarrow |x + 3| = -(x + 3)$

$\therefore |x + 3| - x = -(x + 3) - x = -2x - 3 < -2(-3) - 3 = 6 - 3 = 3 > 2$

$\therefore |x + 3| - x > 2$

$\therefore$ In all possible cases we are getting $|x + 3| - x > 2$.

Parity of an integer:

It is the property of an integer of being even or odd.

e.g.: 10 and 18 numbers have same parity since both are even, 10 and 13 numbers h opposite parity since one is even and other is odd.

Theorem 2.10.1: (parity theorem):

Any two consecutive integers have opposite parity.

Proof:

By division into cases.

Let n and n + 1 are two consecutive integers. We have to show that n and n + 1 have oppo parity.

Case (i) n is even

$\Rightarrow n = 2k$ for some $k \in \mathbb{Z}$

$\Rightarrow n + 1 = 2k + 1$ which is odd

$\therefore$ n is even $\Rightarrow n + 1$ is odd

Case (ii) n is odd

$\Rightarrow n = 2k + 1$ for some $k \in \mathbb{Z}$

$\Rightarrow n + 1 = 2k + 1 + 1 = 2k + 2$

$\Rightarrow n + 1 = 2(k + 1)$ which is even

$\therefore$ n is odd $\Rightarrow$ n + 1 is even

$\therefore$ Consecutive integers have opposite parity.

Theorem 2.10.2:

There is no integer which is both even and odd.

Proof:

Let $x \in \mathbb{Z}$, such that x is even and odd.	Assuming contradiction
$\therefore x = 2k$	by definition of even integer, $k \in \mathbb{Z}$
$\therefore x = 2k' + 1$	by definition of odd integer, $k' \in \mathbb{Z}$

$\therefore 2k = 2k' + 1 \Rightarrow 2k - 2k' = 1 \Rightarrow 2(k - k') = 1$

$\Rightarrow k - k' = \frac{1}{2}$ but $k, k' \in \mathbb{Z} \Rightarrow k - k' \in \mathbb{Z}$ and $\frac{1}{2} \notin \mathbb{Z}$ $\therefore$ $k - k' = \frac{1}{2}$ and $k - k' \in \mathbb{Z}$ but $k - k' = \frac{1}{2} \notin \mathbb{Z}$, which is a contradiction.

$\therefore$ Our assumption is wrong. $\therefore$ No integer is both even and odd.

Illustration 2.10.4:

For any given integers x, y and z, if $x - y$ is even and $y - z$ is even, then what is the parity of $2x - (y + z)$?

Solution:

Given $(x - y)$ and $(y - z)$ both are even. We know that addition of two even integers is even

$\therefore (x - y) + (y - z) = (x - z)$ is even ...(1)

Consider $2x - (y + z) = (x - y) + (x - z)$

$(x - y)$ is even and by (1) $(x - z)$ is even

$2x - (y + z) =$ addition of two even integers which is even.

$\therefore$ parity of $2x - (y + z)$ is even.

Illustration 2.10.5:

Prove that, from any three given natural numbers, we can select two of them with even sum.

Solution:

Let $a, b, c \in \mathbb{N}$ are any three natural numbers.

Case (i): All are odd.
Then odd + odd = even.

Case (ii): All are even.
Then even + even = even.

Case (iii): One even and two odd.
Then take two odd numbers.
odd + odd = even.

Case (iv): One odd and two even.
Then take two even numbers.
even + even = even

$\therefore$ From any three natural numbers we can select two with even sum.

2.11 QUOTIENT-REMAINDER THEOREM:

If n and m are integers and $n > 0$ we can write $m = qn + r$ for integers q and r with $0 \le r < n$ (Seen in Section 2.8)

Here q is called quotient, n is divisor, m is dividend and r is remainder.

Div and Mod functions: Let m be a non negative integer and n be a positive integer, then

m div n = the integer quotient obtained after dividing m by n.

m mod n = the integer remainder obtained after dividing m by n.

e.g., $m = 37, n = 5$

$\therefore 37 = 7(5) + 2$

$\therefore q = 7, r = 2$

$\therefore$ 37 div 5 = 7 and

$\therefore$ 37 mod 5 = 2

Examples:

35 div 7 = 5, 35 mod 7 = 0

8 div 10 = 0, 8 mod 10 = 8

$\therefore$ m mod 2 = 0 if m is even integer $\therefore r = 0$

= 1 if m is odd integer $\therefore r = 1$

Remember r must be less than 2. $\therefore m = 2q$ or $m = 2q + 1$

For any integer m, m mod k are the integers from 0 to $(k - 1)$

mod	forms	remainders
2	$2q, 2q + 1$	0, 1
3	$3q, 3q + 1, 3q + 2$	0, 1, 2
4	$4q, 4q + 1, 4q + 2, 4q + 3$	0, 1, 2, 3
5	$5q, 5q + 1, 5q + 2, 5q + 3, 5q + 4$	0, 1, 2, 3, 4
⋮		
k	$kq, kq + 1, kq + 2, \ldots, kq + (k - 1)$	$0, 1, 2, \ldots, (k - 1)$

Note that: Any integer m can be written in one of the four forms $m = 4q$, $m = 4q + 1$, $m = 4q +$ 2, $m = 4q + 3$.

Illustration 2.11.1:

Prove that the square of any odd integer can be written in the form $8k + 1$, for some integer

Solution:

Let m be an odd integer. By Quotient – Remainder Theorem, m can be written as $m = 4q$ where $q, r \in \mathbb{Z}$ such that $0 \le r < 4$.

$\therefore$ The possible value of r are 0, 1, 2, 3.

$\therefore \quad m = 4q + 0 \quad = 2(2q)$ which is even

$m = 4q + 1 \quad = 2(2q + 1) - 1$ which is odd

$m = 4q + 2 \quad = 2(2q + 1)$ which is even

$m = 4q + 3 \quad = 2(2q + 1) + 1$ which is odd

$\therefore$ We have to take only two cases

$m = 4q + 1$ and $m = 4q + 3$ since m is odd.

Case (i)

$m = 4q + 1$

$\therefore m^2 = (4q + 1)^2 = 16q^2 + 8q + 1 = 8(2q^2 + q) + 1 = 8k + 1$

$\because q \in \mathbb{Z} \Rightarrow k = 2q^2 + q \in \mathbb{Z}$

Case (ii)

$m = 4q + 3$

$\therefore m^2 = (4q + 3)^2 = 16q^2 + 24q + q = 8(2q^2 + 3q + 1) + 1 = 8k + 1$

$\because q \in \mathbb{Z} \Rightarrow k = 2q^2 + 3q + 1 \in \mathbb{Z}$

$\therefore$ If m is odd then $m^2 = 8k + 1$

Illustration 2.11.2:

Prove that $\forall$ x, y $\in \mathbb{Z}$, if x mod 7 = 5 and y mod 7 = 6 then xy mod 7 = 2.

Solution:

x mod 7 = 5 $\Rightarrow x = 7q + 5$ for some $q \in \mathbb{Z}$

y mod 7 = 6 $\Rightarrow y = 7q' + 6$ for some $q' \in \mathbb{Z}$

Consider $xy = (7q + 5)(7q' + 6) = 49qq' + 42q + 35q' + 30$

$= 7(7qq' + 6q + 5q' + 4) + 2$

$\therefore xy = 7K + 2$ where $K = 7qq' + 6q + 5q' + 4$

$\Rightarrow$ xy mod 7 = 2

Illustration 2.11.3:

For x $\in \mathbb{Z}$. If x mod 5 = 4 then 7x mod 5 = ?

Solution:

x mod 5 = 4 $\Rightarrow x = 5q + 4$ for some $q \in \mathbb{Z}$

$\therefore 7x = 7(5q + 4) = 35q + 28 = 5(7q + 5) + 3$

$\therefore 7x = 5K + 3$ where $K = 7q + 5$

$\Rightarrow$ 7x mod 5 = 3

Illustration 2.11.4:

Using quotient remainder theorem $m = nq + r$, $0 \leq r < n$. Prove that the product of any three consecutive integers is divisible by 3. Take n = 3.

Solution:

By Quotient remainder theorem, m can be written in one of the form $m = 3q$, $m = 3q + 1$, $m = 3q + 2$.

Let the three consecutive numbers are (m – 1), m, (m + 1)

$\therefore$ Product of three consecutive numbers $= (m - 1)\, m\, (m + 1)$

$= m^3 - m$

Case (i): $m = 3q$

$\therefore m^3 - m = (3q)^3 - (3q) = 27q^3 - 3q$

$= 3(9q^3 - q)$

Case (ii): $m = 3q + 1$

$\therefore m^3 - m = (3q+1)^3 - (3q+1)$
$= 27q^3 + 27q^2 + 9q + 1 - 3q - 1$
$= 27q^3 + 27q^2 + 6q$
$= 3(9q^3 + 9q^2 + 2q)$

Case (iii): $m = 3q + 2$

$\therefore m^3 - m = (3q+2)^3 - (3q+2)$
$= 27q^3 + 54q^2 + 36q + 8 - 3q - 2$
$= 27q^3 + 54q^2 + 33q + 6$
$= 3(9q^3 + 18q^2 + 11q + 2)$

$\therefore$ In all 3 cases, it is divisible by 3.

EXERCISE 2.5

(1) Proof by division into cases (1 to 4) for $\forall x \in \mathbb{R}$:

(i) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

(ii) $|x + y| \le |x| + |y|$

(2) Product of any two consecutive integers is even.

(3) Prove that $\forall x, y \in \mathbb{Z}$, if $x \bmod 3 = 2$, and $y \bmod 3 = 1$ then $(x + y) \bmod 3 = 0$

(4) Prove that $\forall x \in \mathbb{Z}$, $x^2 - x + 3$ is odd.

(5) Prove that the square of any integer can be written in the form 4k or 4k + 1 for some $k \in \mathbb{Z}$.

(6) For any given integers x, y and z, if (x + y) and (y – z) are odd, then prove that the parity of 2y + (x
is even.

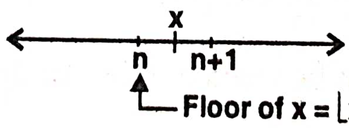
(7) For $x \in \mathbb{Z}$. If $x \bmod 11 = 6$ then show that $4x \bmod 11 = 2$.

2.12 MATHEMATICAL FUNCTIONS FOR COMPUTER SCIENCE: (FLOOR / CEILING)

Floor function: Floor of x is the greatest integer less than or equal to x, denoted by
Therefore we can say that the floor of x "rounds x down".

Ceiling function: Ceiling of x is the smallest integer greater than or equal to x, denoted b
Therefore the ceiling of x "rounds x up".

Note: If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$, otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$.



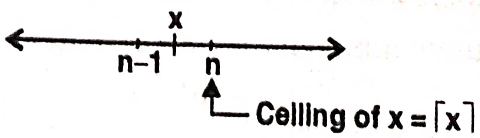


Illustration 2.12.1:

$\lfloor 3.14 \rfloor = 3$ $\lfloor -3.14 \rfloor = -4$ $\lfloor 3 \rfloor = 3$ $\lfloor -3 \rfloor = -3$ $\lfloor \sqrt{3} \rfloor = 1$

$\lceil 3.14 \rceil = 4$ $\lceil -3.14 \rceil = -3$ $\lceil 3 \rceil = 3$ $\lceil -3 \rceil = -3$ $\lceil \sqrt{3} \rceil = 2$

Theorem 2.12.1:

$\forall x \in \mathbb{R}$ and $\forall m \in Z$, $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.

Proof:

Let $\lfloor x \rfloor = n$, then by definition of floor function $n \le x < n + 1$.

Add m to all sides $n + m \le x + m < n + m + 1$

Since $n, m \in Z$

$\therefore$ $n + m \in Z$. Let $n + m = p$

$\therefore$ $p \le x + m < p + 1$

$\therefore$ $\lfloor x + m \rfloor = p$

$\therefore$ $\lfloor x + m \rfloor = n + m$ But $n = \lfloor x \rfloor$

$\therefore$ $\lfloor x + m \rfloor = \lfloor x \rfloor + m$

Illustration 2.12.2:

Let $f(n) = \lfloor n/2 \rfloor + \lfloor n/3 \rfloor$ for $n \in \mathbb{N}$. Find. f (42) and f(35).

Solution:

$\therefore f(42) = \lfloor 42/2 \rfloor + \lfloor 42/3 \rfloor = \lfloor 21 \rfloor + \lfloor 14 \rfloor = 21 + 14 = 35$

$\therefore f(35) = \lfloor 35/2 \rfloor + \lfloor 35/3 \rfloor = \lfloor 17.5 \rfloor + \lfloor 11.67 \rfloor = 17 + 11 = 28.$

Illustration 2.12.3:

Prove that for any non-integer real number x, $\lfloor x \rfloor + \lfloor -x \rfloor = -1$.

Solution:

Let $\lfloor x \rfloor = n$. $\therefore n < x < n + 1$ Since x is non-integer.

$\therefore$ Multiplying both sides by -1 we get

$\therefore -n > -x > -n - 1$

$\therefore -n - 1 < -x < -n$ $\because -n - 1$ is an integer.

$\therefore$ $\lfloor -x \rfloor = -n - 1$

$\therefore \lfloor x \rfloor + \lfloor -x \rfloor = n + (-n - 1) = -1.$

Illustration 2.12.4:

Show that if n is an odd integer. $\lceil n^2/4 \rceil = (n^2 + 3) / 4$.

Solution: Since n is an odd integer therefore for some integer m

$n = 2m + 1$ $\therefore n^2 = (2m + 1)^2$

$n^2 = 4m^2 + 4m + 1$

$n^2 = 4(m^2 + m) + 1$ ... (1)

We know that,

Dividend = Divisor (Quotient) + Remainder.

$\therefore$ It shows that when n^2 is divided by 4 it leaves remainder 1. If 3 is added to this then the new number is divisible by 4 and the new quotient is the next immediate integer which is greater than $n^2 / 4$,

$\therefore \lceil n^2/4 \rceil = (n^2 + 3) / 4.$

Illustration 2.12.5:

How many bytes are required to encode n bits of data where each byte is made up of 8 bits.

(i) n = 500 **(ii)** n = 195

Solution:

The number of required bytes is the smallest integer which is greater than or equal to $\frac{n}{8}$.

(i) $\therefore \left\lceil \frac{500}{8} \right\rceil = \lceil 62.5 \rceil = 63$

(ii) $\therefore \left\lceil \frac{195}{8} \right\rceil = \lceil 24.375 \rceil = 25$

Illustration 2.12.6:

Prove $\lceil x+3 \rceil = \lceil x \rceil + 3$

Solution:

Let $\lceil x \rceil = n$ then by definition of ceiling function $n-1 < x \le n$

$\therefore n-1+3 < x+3 \le n+3$ Adding 3 to all sides

$\therefore n+2 < x+3 \le n+2+1$

$\therefore p < x+3 \le p+1$ Let $p = n+2$

$\therefore \lceil x+3 \rceil = p+1 = n+2+1$

$\therefore \lceil x+3 \rceil = \therefore \lceil x \rceil + 3$

Illustration 2.12.7:

Prove $x^2 - \lfloor x \rfloor^2 < 2\lfloor x \rfloor + 1$

Solution:

Let $\lfloor x \rfloor = n$

$\therefore \quad n \le x < n+1$

$\therefore \quad n^2 \le x^2 < (n+1)^2$

$\therefore \quad x^2 < (n+1)^2$

$\therefore \quad x^2 - \lfloor x \rfloor^2 < (n+1)^2 - \lfloor x \rfloor^2$

$< n^2 + 2n + 1 - n^2$

$< 2n + 1$

$< 2\lfloor x \rfloor + 1$

$\therefore x^2 - \lfloor x \rfloor^2 < 2\lfloor x \rfloor + 1$

Theorem 2.12.2:

If $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, if $q = \lfloor m/n \rfloor$ and $r = m - n\lfloor m/n \rfloor$ then $m = qn + r$ and $0 \le r < n$.

Proof:

Let $m \in m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, $q = \lfloor m/n \rfloor$ and $r = m - n\lfloor m/n \rfloor$

We have to prove that $m = qn + r$ and $0 \le r < n$

Consider $qn + r = \lfloor m/n \rfloor n + m - n\lfloor m/n \rfloor = n\lfloor m/n \rfloor + m - n\lfloor m/n \rfloor = m$

$\therefore m = qn + r$

Now to prove that $0 \le r < n$

Given $q = \lfloor m/n \rfloor$ $\therefore$ By definition of floor function

$q \le \frac{m}{n} < q + 1$

$\therefore qn \le m < qn + n$

$\therefore qn - qn \le m - qn < qn + n - qn$

$\therefore 0 \le m - qn < n$

$\therefore 0 \le r < n$ $\qquad \because m = qn + r \Rightarrow r = m - qn$

Illustration 2.12.8:

Using floor function, find 593 div 13 and 593 mod 13.

Solution:

593 div 13 $= \lfloor 593/13 \rfloor = \lfloor 45.61538461\ldots \rfloor = 45$

593 mod 13 $= 593 - 13 . \lfloor 593/13 \rfloor$

$= 593 - 13(45)$

$= 593 - 585$

$= 8$

Illustration 2.12.9:

Prove that for any odd integer n, $\lfloor n^2/4 \rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)$

Solution:

Let $n = 2k + 1$ for some $k \in \mathbb{Z}$

Consider LHS

$= \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(2k+1)^2}{4}) \right\rfloor = \left\lfloor \frac{4k^2 + 4k + 1}{4} \right\rfloor = \left\lfloor \frac{4(k^2 + k)}{4} + \frac{1}{4} \right\rfloor = \left\lfloor k^2 + k) + \frac{1}{4} \right\rfloor = k^2 + k$

RHS $= \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) = \frac{n^2 - 1}{4} = \frac{(2k+1)^2 - 1}{4} = \frac{4k^2 + 4k + 1 - 1}{4} = k^2 + k$

$\therefore$ LHS = RHS

EXERCISE 2.6

(1) Let f be the mod-10 function. Compute

(a) f (81) (b) f (316) (c) f (1057)

(2) Find (a) $\lfloor 13.2 \rfloor$ (b) $\lceil 13.2 \rceil$ (c) $\lfloor -13.2 \rfloor$ (d) $\lceil -13.2 \rceil$ (e) $\lfloor -0.17 \rfloor$

(f) $\lceil -0.17 \rceil$ (g) $\lfloor 13 \rfloor$ (h) $\lceil 13 \rceil$

(3) For $x \in \mathbb{R}$, show that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$.

(4) How many bytes are required to encode n bits of data where each byte is made up of 8 bits.

(i) n = 253 (ii) n = 1017 (iii) n = 680

(5) Prove that for any odd integer n, $\lceil n^2/4 \rceil = \frac{n^2 + 3}{4}$.

ANSWERS 2.6

(1) (a)1, (b) 6 (c) 7

(2) (a)13 (b) 14 (c) –14 (d) –13 (e) –1 (f) 0 (g) 13 (h) 13

(4) (i)32 (ii) 128 (iii) 85

2.13 INDIRECT ARGUMENTS: (TWO-CLASSICAL THEOREMS)

The following two famous theorems of mathematics are the examples of indirect argumen

Theorem 2.13.1: (Classical Theorem)

Prove that $\sqrt{2}$ is irrational.

Proof:

Let us assume that $\sqrt{2}$ is rational.

$\therefore \exists$ a, b $\in \mathbb{Z}$, b $\neq 0$, and no common factor between a and b such that $\sqrt{2} = \frac{a}{b}$

$\therefore 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ is even $\Rightarrow$ a is even i.e. $a = 2k$ for $k \in \mathbb{Z}$

$\therefore 2b^2 = a^2 = (2k)^2 \Rightarrow 2b^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b^2$ is even

$\Rightarrow$ b is even

Now a and b are both even

$\Rightarrow$ 2 is common factor of a and b which contradicts the fact that a and b have no com factor.

$\therefore$ Our assumption $\sqrt{2}$ is rational is false.

$\therefore \sqrt{2}$ is irrational.

Theorem 2.13.2 (Classical Theorem):

There are infinitely many prime numbers

Proof:

Let as assume that there are finitely many primes.

Let $P_1 = 2 < P_2 = 3 < P_3 = 5 < \ldots < P_n$ are all the primes. Now construct a new number P as $P = P_1.P_2\ldots P_n + 1$ clearly $P > P_i$ for $i = 1, 2, \ldots, n$

Since $P_1, P_2, \ldots P_n$ are all primes P cannot be a prime

$\therefore$ P must be divisible by at least one of P_i for $i = 1, 2, \ldots, n$

Let us say P is divisible by P_m $1 \leq m \leq n$

$\therefore P = (P_1, P_2, \ldots P_n)(P_m) + 1$. (By Quotient-Remainder Theorem)

$\therefore$ When P is divided by P_m we get 1 as remainder

i.e. P_m is not a prime number from $P_1, P_2, \ldots P_n$ which is contradiction to our assumptio $P_1, P_2, \ldots P_n$ are all the primes.

$\therefore$ There are infinitely many primes.

Illustration 2.13.1:

Prove that $3 + 5\sqrt{2}$ is irrational.

Solution:

Let us assume that $3 + 5\sqrt{2}$ is rational.

$\therefore 3 + 5\sqrt{2} = \frac{a}{b}$ for some a, b $\in \mathbb{Z}$, b $\neq 0$

$\therefore 5\sqrt{2} = \frac{a}{b} - 3$

$\therefore 5\sqrt{2} = \frac{a - 3b}{b}$

$\therefore \sqrt{2} = \frac{a-3b}{5b}$ $\because a, b \in \mathbb{Z} \Rightarrow a - 3b \in \mathbb{Z}$ and $b \neq 0 \Rightarrow 5b \neq 0 \in \mathbb{Z}$

Let $x = a - 3b$ and $y = 5b$

$\therefore \sqrt{2} = \frac{x}{y}$

$\Rightarrow \sqrt{2}$ is rational number by definition of rational number.

This contradicts the fact that $\sqrt{2}$ is irrational.

$\therefore$ Our assumption was wrong.

$\therefore 3 + 5\sqrt{2}$ is irrational.

2.14 APPLICATIONS IN ALGORITHMS:

An algorithm is a step by step procedure for solving a problem. It is described by an input data, output data, preconditions (specify restrictions on input data) and post conditions (specify the result).

Properties of algorithm:

It must be

- Finite
- Executable
- Unambiguous
- 0 or more input
- 1 or more output
- General (can solve a particular type of problem for which it is written)

While writing an algorithm the following information can be included:

- Name of the algorithm
- List of input and output variables with their names and data types.
- Description of working of algorithm.
- The step by step procedure of the algorithm with explanation wherever required.

Algorithms can be transformed into program using any high level language.

Division Algorithm:

Illustration 2.14.1:

Given m a non negative integer and n a positive integer, write an algorithm to find integers q and r such that $m = qn + r, 0 \le r < n$

Solution:

Division Algorithm

Input m and n as given type

Step 1: $r = m, q = 0$

Step 2: while $(r \ge n)$ (loop will continue till $r \ge n$. It will stop when r becomes less than n repeatedly subtract n from r)

$r = r - n$

$q = q + 1$ (Add 1 to q for each iteration)

end while

(After finishing execution of the loop, $m = qn + r$)

Output: q, r (non negative integers)

Euclidean Algorithm:

Illustration 2.14.2

Given two integers $m > n \geq 0$. Write an algorithm to calculate gcd(m, n).

Solution:

Euclidean Algorithm

Input: m and n as given

Step 1: $r = n$

Step 2: while $(n \neq 0)$

$r = m \bmod n$ (This value can be calculated by calling the division algo

$m = n$

$n = r$

end while

$gcd = m$

Output: gcd

Illustration 2.14.3

Use division algorithm for input $m = 21, n = 4$ to find quotient q and remainder r.

Solution:

Input: $m = 21, n = 4$

Step 1: $r = 21, q = 0$

Step 2: while $(r = 21 \geq n = 4)$

Condition	T/F	Iteration No.	$r = r - n$	$q = q + 1$
$21 \geq 4$	T	1	$21 - 4 = 17$	$0 + 1 = 1$
$17 \geq 4$	T	2	$17 - 4 = 13$	$1 + 1 = 2$
$13 \geq 4$	T	3	$13 - 4 = 9$	$2 + 1 = 3$
$9 \geq 4$	T	4	$9 - 4 = 5$	$3 + 1 = 4$
$5 \geq 4$	T	5	$5 - 4 = 1$	$4 + 1 = 5$
$1 \geq 4$	F	Exit Loop		

Output: $q = 5, r = 1$

$\because 21 = 5 \times 4 + 1$

Illustration 2.14.4

Use Euclidean Algorithm to find gcd, for inputs $m = 105, n = 90$.

Solution:

Input: $m = 105, n = 90$

Step 1: $r = 90$

Step 2: while $(90 = n \neq 0)$

Condition	T/F	Iteration No.	$r = m \bmod n$	$m = n$	$n = r$
$90 \neq 0$	T	1	15	90	15
$15 \neq 0$	T	2	0	15	0
$0 \neq 0$	F	Exit Loop			

Output: gcd = m = 15

∴ gcd (105, 90) = 15

Illustration 2.14.5

Use division algorithm to find q and r. Also use Euclidean algorithm to find gcd for m = 72, n = 27.

Solution:

Division Algorithm

Condition	T/F	Iteration No.	r = r − n	q = q + 1
$72 \geq 27$	T	1	45	1
$45 \geq 27$	T	2	18	2
$18 \geq 27$	F	Exit Loop		

$q = 2, r = 18 \therefore 72 = 2 \times 27 + 18$

Euclidean Algorithm

Condition	T/F	Iteration No.	r = m mod n	m = n	n = r
$27 \neq 0$	T	1	18	27	18
$18 \neq 0$	T	2	9	18	9
$9 \neq 0$	T	3	0	9	0
$0 \neq 0$	F	Exit Loop			

∴ gcd = m = 9 ∴ gcd(72, 27) = 9

EXERCISE 2.7

1) Use division algorithm for the following inputs to find quotient q and remainder r.
   (i) m = 500, n= 17 (ii) m = 1015, n = 39 (iii) m = 5219, n = 87
2) Use Euclidean algorithm to find gcd of following pairs.
   (i) m = 330, n = 156 (ii) m = 90, n = 45 (iii) m = 1188, n = 385 (iv) m = 40, n = 21

ANSWERS 2.7

1) (i) q = 29, r = 7 (ii) q = 26, r = 1 (iii) q = 59, r = 86
2) (i) 6 (ii) 45 (iii) 11 (iv) 1

ASSIGNMENT 2

1) Find the negations of the following.
   (i) $P \wedge (q \wedge r)$
   (ii) $(p \leftrightarrow q) \vee (\sim q \rightarrow r)$
   (iii) $(p \rightarrow q) \wedge r$
2) Determine the truth value of each of the following statements where U = {1, 2, 3} is the universal set.
   (a) $\exists x \, \forall y, x^2 < y + 1$ (b) $\forall x \, \exists y, x^2 + y^2 < 12$ (c) $\forall x \, \forall y, x^2 + y^2 < 12$
3) Negate each of the following statements.
   (a) $\exists x \in A, x + 4 = 11$ (b) $\forall x \in A, x + 4 < 11$
   (c) $\exists x \in A, x + 2 < 4$ (d) $\forall x \in A, x + 4 \leq 7$.
   Where A = {1, 2, 3, 4, 5}.

(4) Convert into logical form and then negate the following statements.

(i) Everybody likes somebody.

(ii) All even integers are divisible by 2.

(iii) Every natural number is an integer.

(iv) Some healthy can fly.

(5) Convert the following arguments using quantifiers:

(i) All healthy people are vegetarian.
Mona is not healthy.
∴ Mona is not vegetarian.

(ii) All healthy people do exercise every day.
Ruby does exercise every day.
∴ Ruby is a healthy person.

(iii) No good car is cheap.
Ford is not cheap.
∴ Ford is a good car.

(6) Prove directly that:

(i) The difference of odd and even integer is odd.

(ii) The difference of any two rational numbers is a rational number.

(iii) If $r, s \in \mathbb{Z}$ such that r is even and s is odd then $r^2 + 3s$ is odd.

(iv) If $r \in \mathbb{Z}$ is an odd integer than $a^2 + a$ is even.

(7) Prove by contraposition proof or contradiction proof.

(i) The sum of any rational and any irrational number is irrational.

(ii) $\forall x \in \mathbb{Z}$, if x^2 is even then x is even.

(iii) If $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2$, then at least one of a and b is even.

(iv) $\forall x, y \in \mathbb{Z}$, if $x + y$ is even then either both x and y are even or both are odd.

(v) For $x, y \in \mathbb{Z}$, if $x + y < 50$ then either $x < 25$ or $y < 25$

(vi) $6 - 7\sqrt{2}$ is irrational.

(vii) $\sqrt{2} + \sqrt{3}$ is irrational.

(8) Prove by proof by cases:

(i) Every $x \in \mathbb{Z}$, $x^4 = 8n$ or $x^4 = 8n + 1$

(ii) Every $x \in \mathbb{Z}$, s.t. $x = n^3$ then x is multiple of 9 or 1 less than multiple of 9 or 1 more than mul 9.
(Hint: Take $n = 3p, 3p - 1, 3p - 2$)

(iii) $\forall x, y \in \mathbb{Z}$, $x + y$ and $x - y$ both are either even or both are odd.
(Hint: Take 3 cases, x, y both even, both odd and one even and one odd)

(iv) $\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil = x \ \forall x \in \mathbb{Z}$. (Hint: Take 2 cases x is odd, x is even)

(9) Use Quotient remainder theorem to prove that the product of any two consecutive integers form 3p or 3p + 2 for some $k \in \mathbb{Z}$. Take n = 3.

(10) Prove that $\forall x \in \mathbb{Z}$, (a) if x mod 5 = 3 then x^2 mod 5 = 4; (b) if x mod 7 = 6 then (5x – 3) mod 7 = 6

(11) For any given integers x, y and z, if $(x - y)$ is odd and $(y - z)$ is even, then what is the parity of $x + y - 2z$?

(12) Prove that $m(m^2 - 1)(m + 2)$ is divisible by 4 for any $m \in \mathbb{Z}$.

(13) Total number of passengers = 30 and capacity of each vehicle = 7

(i) What is the maximum number of vehicles required, if each will send with full capacity?
(Hint: Use floor function)

(ii) How many vehicles are required if one vehicle is partially filled?
(Hint: Use ceiling function)